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FOUNDATIONS, THEORY OF SETS, LOGIC

Rožanskaya, Yu. A. On the equivalence of two definitions of completeness of a system of axioms of A. Kolmogorov and H. Weyl. Moskov Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 197–198. (Russian)

The definitions are: (α) The axiom system S is complete if it becomes inconsistent by the adjunction of any independent axiom, expressed in the fundamental notions of S. (β) The axiom system S is complete, if any two models of S are isomorphic. The author proves directly the equivalence of the negations $\neg(\alpha)$ and $\neg(\beta)$. If $\neg(\alpha)$, then there is an axiom B, so that the systems $S+B$ and $S+\neg B$ are consistent, and thus both of them have models which are unisomorphic models of S. Now let M be the intended interpretation of S, and assume that S admits two non-isomorphic models L and L'. If L and L' have different cardinals, an axiom about the cardinal of M can be adjoined to S, disproving (α). Let L and L' have the same cardinal and let φ be a one-to-one mapping between L and L', and let X be a relation, expressible in S. φ is called an X-equivalence, if, for every pair a, b of elements of L, $X(a, b)$ entails $X(\varphi a, \varphi b)$. For every φ there exists an X so that φ is not an X-equivalence. Choose a particular φ and a particular χ with this property. Thus the following axiom, independent of S, can be adjoined to S, giving a consistent system: (C) Every one-to-one mapping of L onto M is an X-equivalence. *A. Heyting.*

Schmidt, Jürgen. Mehrstufige Austauschstrukturen. Z. Math. Logik Grundlagen Math. **2** (1956), 233–249.

E: Grundmenge. M, A_1, A_2, F: Teilmengen von E. $\mathfrak{F}$: System aller endlichen Teilmengen von E. $\emptyset$: leere Teilmenge von E. $\mathbf{C}$ (mehrstufiger Hüllenoperator oder Stufenoperator): Abbildung der Potenzmenge $\mathfrak{P}E$ in sich selbst, die folgenden Axiomen genügt: ($\mathbf{C}1$) $M \subseteq \mathbf{C}M$, ($\mathbf{C}2$) $(M_1 \subseteq M_2) \Rightarrow (\mathbf{C}M_1 \subseteq \mathbf{C}M_2)$. Ableitung $\mathbf{D}M$ der Menge M: $(x \in \mathbf{D}M) \Leftrightarrow (x \in \mathbf{C}(M - M \cap \{x\}))$. System $\mathfrak{U} = \mathfrak{U}_{\mathbf{C}}$ der unabhängigen Mengen: $(U \in \mathfrak{U}) \Leftrightarrow (U \cap \mathbf{D}U = \emptyset)$. $\mathfrak{U}^* = \mathfrak{U} \cap \mathfrak{F}$. Einstufigkeitsaxiom: $\mathbf{C}\mathbf{C}M = \mathbf{C}M$. Endlichkeitsaxiom: $\mathbf{C}M = \mathbf{U}\mathbf{C}F$, wobei $F \subseteq M$, $F \in \mathfrak{U}^*$. Austauschaxiom:

$$((F \in \mathfrak{U}^*) \wedge (y \in \mathbf{C}(F \cup \{x\})) \wedge (y \notin \mathbf{C}F)) \Rightarrow (x \in \mathbf{C}(F \cup \{y\})).$$

Ein Stufenoperator, der dem Endlichkeits-, sowie dem Austauschaxiom genügt, wird ein „(mehrstufiger) Austauschoperator" genannt. H(aupt)-N(öbeling)-P(auc)-Axiom:

$$(((F \cup \{x\}) \in \mathfrak{U}^*) \wedge ((F \cup \{x, y\}) \notin \mathfrak{U}^*) \wedge ((F \cup \{x, z\}) \notin \mathfrak{U}^*) \wedge (y \neq z)) \Rightarrow ((F \cup \{y, z\}) \notin \mathfrak{U}^*).$$

Durchschnittsaxiom:

$$(((A_1 \cup A_2) \in \mathfrak{U}^*) \wedge ((A_1 \cup \{x\}) \notin \mathfrak{U}^*) \wedge ((A_2 \cup \{x\}) \notin \mathfrak{U}^*)) \Rightarrow (((A_1 \cap A_2) \cup \{x\}) \notin \mathfrak{U}^*).$$

Basis-Existenzaxiom: Ist F eine Menge aus $\mathfrak{U}^*$ und $x \in \mathbf{C}F$, so gibt es unter denjenigen Mengen $A \subseteq F$ von denen x abhängt (d.h. $x \in \mathbf{C}A$) gewiss eine kleinste (die „Basis von x in F"). Satz: Ist $\mathbf{C}$ ein Austauschoperator, dann sind das HNP-, das Durchschnitts-, das Basis-Existenz-, und das Einstufigkeitsaxiom äquivalent. *C. Pauc* (Nantes).

Callahan, Francis P.; and Kneale, Samuel G. A note on the Schroeder-Bernstein theorem. Amer. Math. Monthly **64** (1957), 423–424.

The authors consider the Schroeder-Bernstein theorem which states that, given two sets, A and B, and two one-to-one mappings f and g such that $f(A) \subset B$ and $g(B) \subset A$, then there is a one-to-one mapping F which maps A onto all of B. In the classical proof of this theorem, F is chosen so that $F = f$ on A_f and $F = g^{-1}$ on $A_{g^{-1}}$, where A is divided into two parts A_f and $A_{g^{-1}}$. It is shown that if F is to satisfy the theorem, then the choice of A_f is inevitable up to the possible addition of a set invariant under the transformation gf. To show this, the authors provide necessary and sufficient conditions that F satisfy the conditions of the theorem, as follows: (1) g^{-1} must exist for all points of $A_{g^{-1}}$, so that $g(B) \subset A_{g^{-1}}$; (2) $A = A_f + A_{g^{-1}}$; and (3) $B = F(A_f) + F(A_{g^{-1}}) = fA_f + g^{-1}A_{g^{-1}}$, where $+$ indicates that the summands are disjoint. *E. J. Cogan.*

Sierpinski, W. Sur un problème concernant les suites infinies d'ensembles. Rend. Sem. Mat. Messina **1** (1955), 143–145.

Let S be a sequence of arbitrary sets, and S_σ the family of all sets expressible as unions of terms of S. It is clear that the cardinal of S_σ is at most c, and that, for suitable S, it can be exactly c, or any cardinal $\leq \aleph_0$. The author proves, without recourse to the continuum hypothesis, that the cardinal in question can, in fact, be any cardinal $\leq c$. *L. Gillman* (Lafayette, Ind.).

Wyler, Oswald. On an axiom of Bourbaki. Proc. Amer. Math. Soc. **8** (1957), 672.

The author derives Bourbaki's axiom A2 [Théorie des ensembles, Ch. I, II, Actualités Sci. Ind., no. 1212, Hermann, Paris, 1954; MR **16**, 454]: $(\forall x)(\forall y)\ \mathrm{Coll}_z\ (z = x \text{ or } z = y)$ from the weaker axiom $(\forall x)\ \mathrm{Coll}_y\ (y = x)$. *L. Gillman* (Lafayette, Ind.).

Schmidt, Jürgen. Einfacher, ordinalzahlfreier Beweis für die Wohlordnung der Mächtigkeiten. Math. Z. **67** (1957), 299–302.

The well-ordering of cardinals is derived from the maximal principle. *L. Gillman* (Lafayette, Ind.).

Ginsburg, Seymour. On mappings from the family of well ordered subsets of a set. Pacific J. Math. **6** (1956), 583–589.

For an ordered set E let ωE be the system of all well ordered subsets of E ordered by the relation "is an initial

segment of". Any strictly increasing function from ωE to E is called a k-function. If there exists a k-function on E (i.e., from ωE to E), then E is called a k-set; otherwise, E is called a k'-set. E.g., the sets R of rationals and R^+ of reals are k'-sets [Kurepa, Bull. Internat. Acad. Yougoslav. Cl. Sci. Math. Phys. Tech. (N.S.) **12** (1954), 35–42; MR **16**, 1006]; this is a special case of the fact (Th. 10) that each simply ordered infinite group is a k'-set. Th. 8: If E, F are k-sets, then so is $E \times F$. Do there exist k'-sets E, F such that $E \times F$ is a k-set? Th. 7: Every chain is a terminal section of some k-set. Th. 5: The ordered sum, over the dual of any ordinal, of k-sets is a k-set. Th. 1: If f is any k-function on E, then for every $X \in \omega E$ there exists an $x \in X$ such that $f(X) \leqq x$. *Đ. Kurepa* (Zagreb).

Goodstein, R. L. Logic-free formalisations of recursive arithmetic. Math. Scand. **2** (1954), 247–261.

In this formal system the only axioms are primitive recursive function definitions, substitution rules for equations, and the uniqueness rule

$$F(Sx)=H(x, F(x)) \vdash F(x)=H^x F(0)$$

where $H^0 t=t$, $H^{Sx}t=H(x, H^x t)$. With the usual notations, $|a, b|$ denoting the symmetric difference, implication is introduced by $(F=G)\rightarrow(A=B)$ meaning $\{1 \dot{-} |F, G|\}|A,B|=0$. A deduction theorem for this implication is valid. Some alternatives for the uniqueness rule are studied.

A. Heyting (Amsterdam).

Rieger, Ladislav. On a fundamental theorem of mathematical logic. Časopis Pěst. Mat. **80** (1955), 217–231. (Czech. Russian and English summaries)

Let A be a Boolean algebra and Φ a system of infinite sequences $\{a_k\}$ $(k=1, 2, \cdots)$ of elements a_k of A. Then A is called a Φ-algebra under the following conditions: a) if $\{a_k\} \in \Phi$, then $\{a_k'\} \in \Phi$, the prime indicating complementation; b) if $\{a_k\} \in \Phi$, then $\{a \cup a_k\} \in \Phi$ for every $a \in A$; c) the system Φ contains every infinite sequence of elements of A all of whose terms, except for a finite number, are constant (the so-called trivial sequences); d) if $\{a_k\} \in \Phi$, then there exists in A a least upper bound $\bigcup a_k$ of the elements a_k $(k=1, 2, \cdots)$. In the Φ-algebra for each sequence $\{a_k\} \in \Phi$ there also exists a greatest lower bound $\bigcap a_k$. The concept of a Φ-algebra is a simplification of the concept of a generalized σ-algebra introduced earlier by the author [Czechoslovak Math. J. **1(76)** (1951), 29–40; MR **14**, 347]. A non-empty set $I \subset A$ is called a Φ-ideal under the following conditions: a) $I \neq A$; b) if $\{a_k\} \in \Phi$ and $a_k \in I$ for $k=1, 2, \cdots$, then $\bigcap a_k \in I$; c) if $a \in I$, $a \leqq b$, then $b \in I$. A Φ-ideal I is called a prime ideal if the following condition is also fulfilled: d) if $\{a_k\} \in \Phi$ and $\bigcup a_k \in I$, then there exists an index k such that $a_k \in I$.

The fundamental theorem is proved by simple induction: If A is a Φ-algebra with a countable system Φ of sequences, then for every Φ-ideal $I \subset A$ there exists a prime Φ-ideal $P \supset I$.

With reference to the Hilbert-Ackermann system a construction is sketched for the well-known Boolean algebra consisting of classes of equivalent formulas in the lower predicate calculus. The author calls it the algebra of Lindenbaum. It is also shown, as in the article cited above, that the algebra of Lindenbaum may be considered as a Φ-algebra. In this case Φ consists of trivial sequences of classes and of sequences of the form $\{[\mathfrak{A}^*(x_k)]\}$, where $[\mathfrak{A}^*(x_k)]$ denotes the class of formulas equivalent to the formula $\mathfrak{A}^*(x_k)$, and $\mathfrak{A}^*(x_k)$ is derived from the formula $\mathfrak{A}^*(x_1)$ containing the free variable x_1 by the replacement of x_1 by x_k and the allowable renaming of the bound variables entering in $\mathfrak{A}(x_1)$.

It is shown that every elementary consistent theory can be considered as a Φ-ideal in the Lindenbaum Φ-algebra and every complete elementary consistent theory as a prime Φ-ideal (in agreement with Tarski). The fundamental theorem of the author is simply an algebraic formulation of the so-called theorem of Lindenbaum on the possibility of extending an elementary consistent theory to a complete consistent theory. It is stated without proof that by means of the fundamental theorem it is also possible to prove without difficulty the theorem of Loomis on σ-algebras as homomorphic images of σ-fields of sets and the theorem on the freedom of the σ-field of Borel sets of the Cantor discontinuum. As a corollary of the Lindenbaum theory a proof is given of the Gödel completeness theorem. Also the proof of the Skolem-Löwenheim theorem proceeds in a completely analogous way. In conclusion several well-known methods are mentioned for proof of the Gödel completeness theorem and, as the author mentions, of the equivalent theorem of Lindenbaum. It would be desirable to clarify the interdependence of these methods.

In the article mention is also made of the possibility of making use in the theory of probability of the representation of the Lindenbaum algebra by means of the Borel sets in the Cantor discontinuum.

Jiří Bečvář (RŽMat **1956**, no. 968).

Rasiowa, Helena; and Mostowski, Andrzej. A geometric interpretation of logical formulae. Studia Logica **1** (1953), 254–275 (1954). (Polish. Russian and English summaries)

The article contains a geometric interpretation of quantors in the classical and in the intuitionistic predicate calculus and demonstrates the equivalence of this interpretation with the well-known algebraic interpretation [for the intuitionistic calculus the interpretation is given in the article by Mostowski in J. Symb. Logic **13** (1948), 204–207; MR **10**, 421]. This result implies on the basis of well-known theorems about the adequacy of the algebraic interpretation [Rasiowa and Sikorski, Fund. Math. **40** (1953), 62–95; MR **15**, 668] that the geometric interpretation is adequate. The essential relation which permits us to pass from the algebraic interpretation of logical formulae to the geometric interpretation given by the authors is expressed by the fact (not mentioned by them) that the sum $\sum_{t \in T} X_t$ of the subsets X_t of the space $\mathfrak{X}$ is given by the projection on the $\mathfrak{X}$-axis of the set of all points $\langle t, x \rangle$, where $x \in X_t$. *J. Łoś* (RŽMat **1956**, no. 6334).

Wright, Fred B. Ideals in a polyadic algebra. Proc. Amer. Math. Soc. **8** (1957), 544–546.

The analysis and application of some of the ideas of P. Halmos' papers on polyadic algebras [Compositio Math. **12** (1955), 217–249; Amer. Math. Monthly **63** (1956), 363–387; Fund. Math. **43** (1956), 255–325; MR **17**, 1172; **19**, 112] leads the author to related results from which the semi-simplicity of a polyadic algebra follows as a corollary. It is shown (i) that there exists an inclusion-preserving one-to-one correspondence between the polyadic ideals of a polyadic algebra A and the ideals of the Boolean algebra of closed elements of A; and (ii) that every ideal M of a polyadic algebra contains a polyadic ideal which is maximal in M. *A. Robinson.*

Rose, T. A. The contradictory function. Mind **66** (1957), 331–350.

ALGEBRA

Combinatorial Analysis

Tietze, Heinrich. Über Schachturnier-Tabellen. Math. Z. **67** (1957), 188–202.

It is shown that if n is odd then, up to isomorphism, there is only one tournament schedule for a round robin chess tournament involving a minimal number of rounds in which all players alternate in playing the white and black pieces. If n is even and the schedule involves a minimal number of rounds then some players will have to play the same color in at least two consecutive games. In this case there are several essentially different ways of scheduling the games so that no player plays the same color in two consecutive games more than once, but the problem of finding the total number of non-isomorphic schedules satisfying this condition is left unsolved.

L. Moser (Edmonton, Alta.).

Balliccioni, M. A. Calcul symbolique: une application du théorème du produit. Bull. Soc. Roy. Sci. Liège **25** (1956), 570–572.

Using the convolution theorem and various relations involving the Gamma function and Euler's constant γ, the author obtains the following two identities, where C_m^r are the usual binomial coefficients:

$$\sum_{r=0}^{m} C_m^r \frac{(-1)^r}{r+1} = \frac{1}{m+1}, \quad \sum_{r=0}^{m} C_m^r \frac{(-1)^r}{(r+1)^2} = \frac{1}{m+1} \sum_{r=0}^{m} \frac{1}{r+1}.$$

The first of these could also be obtained directly by expanding the left side of the identity $[(-1)+1]^{m+1}=0$, transposing the first term, and changing the index of summation, after dividing both sides by $m+1$.

D. L. Bernstein (Los Angeles, Calif.).

Gould, H. W. Final analysis of Vandermonde's convolution. Amer. Math. Monthly **64** (1957), 409–415.

Numerous combinatorial identities have been obtained by deriving relations which the coefficients in series expansions must satisfy. From an expansion valid for integral exponents one derives a polynomial identity true for all real values of the variables. Frequently tricks have been employed to obtain the expansions to be used in this way. After a review of several historical examples of the procedure described there is given the "generalized difference series expansion"

$$(*)\quad f(\alpha+y)=\sum_{n=0}^{\infty}\binom{(\alpha+\beta n)/z}{n}\frac{\alpha}{\alpha+\beta n}\, z^n \Delta_{iz}^n f(t+y)\Big]_{t=-\beta n}$$

where $\Delta_{iz}^n f(t)=[(-1)^n/z^n]\sum_{i=0}^{n}(-1)^i\binom{n}{i} f(t+iz)$. This includes as special or limiting cases all the series expansions which have been so used. A limiting case is the "generalized differential series"

$$f(\alpha)=\sum_{n=0}^{\infty} B_n(\alpha,\beta) f^{(n)}(t)]_{t=-\beta n}$$

where $B_n(\alpha,\beta)=\alpha(\alpha+\beta n)^{n-1}/n!$. It is pointed out that not much detailed convergence information is available about the expansion (*) in the general case. *T. N. E. Greville.*

Ryser, H. J. Combinatorial properties of matrices of zeros and ones. Canad. J. Math. **9** (1957), 371–377.

The first main result of this paper settles the existence problem for the class of m by n matrices all of whose entries are zeros and ones with specified row and column sums. Let $R=(r_1, \cdots, r_m)$, where r_i $(i=1, \cdots, m)$ is the sum of the ith row, and $S=(s_1, \cdots, s_n)$, where s_j $(j=1, \cdots, n)$ is the sum of the jth column. If t is the total number of 1's in the matrix, then trivially $t=r_1+\cdots+r_m=s_1+\cdots+s_n$. Let us regard $r_1+\cdots+r_m$ as a partition of t and construct the dual partition $t=u_1+u_2+\cdots+u_k+\cdots+u_n$, i.e., where u_k is the number of r's greater than or equal to k (trivially no r exceeds n). The author shows that a matrix with row sums $R=(r_1, \cdots, r_m)$ and column sums $S=(s_1, \cdots, s_n)$ exists if and only if, the s's being renumbered so that $s_1\geq\cdots\geq s_n$, the following conditions hold: $s_1+\cdots+s_i\leq u_1+\cdots+u_i$ $(i=1, \cdots, n-1)$, $s_1+\cdots+s_n=u_1+\cdots+u_n=t$.

The second main result of this paper is the demonstration that any m by n zero-one-matrix with row sums $R=(r_1, \cdots, r_m)$ and column sums $S=(s_1, \cdots, s_n)$ can be transformed into any other by a succession of a finite number of interchanges, an interchange being the replacement of a minor of one of the forms $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ by the other.

Marshall Hall, Jr.

Harary, Frank. The number of dissimilar supergraphs of a linear graph. Pacific J. Math. **7** (1957), 903–911.

This is one of a series of papers on enumeration theorems for graphs [Trans. Amer. Math. Soc. **78** (1955), 445–463; Pacific J. Math. **6** (1956), 57–64; MR **16**, 844; **18**, 56]. A formula is obtained for the number of dissimilar ways in which a given graph H can be exhibited as a subgraph of a (p, q)-graph, i.e., a graph with p vertices and q edges. Special consideration is given to the cases in which H is (i) a complete graph and (ii) a circuit. In particular the formula gives the number of dissimilar Hamiltonian circuits to be found among all the (p, q)-graphs.

W. T. Tutte (Toronto, Ont.).

See also: Algebraic Topology: Kelly; Sedláček; Collatz und Sinogowitz; Kotzig. **Computing Machines:** Lloyd. **Statistics:** Roy and Laha; Freeman; Lafon (2 articles); Thompson; Kempthorne. **Economics, Management Science:** Supnick.

Linear Algebra

Burger, E. Eine Bemerkung über nicht-negative Matrizen. Z. Angew. Math. Mech. **37** (1957), 227.

Gonçalves, Vicente J. Sur la transformée J d'une matrice carrée. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 343–352.

Elégant exposé de la réduction des matrices carrées aux matrices canoniques de Jordan.

G. Papy.

Parodi, Maurice. Sur une méthode de localisation des valeurs caractéristiques de certaines matrices. C. R. Acad. Sci. Paris **244** (1957), 1597–1598.

The theorem of Gersgorin [Izv. Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1931**, 749–754], based upon earlier results of L. Levy [C. R. Acad. Sci. Paris **93** (1881), 706–708] and others, states that each characteristic root of a complex matrix lies inside the union of certain circles in

the complex plane. For certain matrices of a somewhat specialised type the characteristic roots also lie outside the intersection of another set of circles.

D. E. Rutherford (St. Andrews).

See also: Foundations, Theory of Sets, Logic: Schmidt. **Combinatorial Analysis:** Ryser. **Groups and Generalizations:** Gravett; Banaschewski; Mahler; Landin and Reiner. **Topological Vector Spaces:** Fan. **Algebraic Topology:** Collatz und Sinogowitz. **Numerical Methods:** Červená. **Fluid Mechanics, Acoustics:** Kampé de Fériet. **Programming, Resource Allocation, Games:** Pearson.

Polynomials

See: Fields, Rings: Amitsur. **Geometry of Numbers:** Turán. **Functions of Complex Variables:** Specht; Parodi.

Partial Order, Lattices

Szász, G. On relatively complemented lattices. Acta Sci. Math. Szeged **18** (1957), 48–51.

This note is concerned with the relation between complements and relative complements in a lattice. A sample result is the following converse of a theorem of von Neumann. Let L be a lattice with zero 0 and unit i; suppose $a \leqq r \leqq b$ in L; assume r has a relative complement s in the interval $[a, b]$; if y is a relative complement of a in $[0, s]$, z a relative complement of b in $[s, i]$, and t a relative complement of s in $[y, z]$, then t is a complement of r in L and $(a \cup t) \cap b = s = a \cup (t \cap b)$; moreover, every t with these properties can be obtained in this way.

R. S. Pierce (Seattle, Wash.).

Bialynicki-Birula, A.; and Rasiowa, H. On the representation of quasi-Boolean algebras. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 259–261, XXII. (Russian summary)

An abstract algebra $A\langle +, \cdot, \sim\rangle$ is called a quasi-Boolean algebra if $A\langle +, \cdot\rangle$ is a distributive lattice with a zero element 0 and a unit element 1 and if $\sim$ is a unary operation satisfying the conditions: $\sim\sim a = a$, $\sim(a \cdot b) = \sim a + \sim b$ (and consequently $\sim(a+b) = \sim a \cdot \sim b$, $\sim 0 = 1$ and $\sim 1 = 0$) for any $a, b \in A$. Let $\mathfrak{X}$ be a set of elements and let g be a permutation of $\mathfrak{X}$ with $g = g^{-1}$, and let, for every subset X of $\mathfrak{X}$, $\sim X$ be defined by $\sim X = \mathfrak{X} - g(X)$. Every family of subsets of $\mathfrak{X}$ closed under the set-theoretic operations of sum and product and under the operation $\sim$ is called a quasi-field of sets and it is easy to show that every quasi-field is a quasi-Boolean algebra. The authors show, using the method of Stone's proof of the representation of Boolean algebras as fields of sets [Trans. Amer. Math. Soc. **40** (1936), 37–111], that every quasi-Boolean algebra is isomorphic to a quasi-field of sets. Let q be a prime filter (dual prime ideal) of A and let $\tilde{q} = \{\sim x, x \in q\}$; then $\tilde{q}$ is a prime ideal of A and the set $g(x) = A - \tilde{q}$ is a prime filter. Let, for any subset X of the set $\mathfrak{X}$ of all prime filters of A, $\sim X$ be defined by $\sim X = \mathfrak{X} - g(X)$; then the family of all subsets of $\mathfrak{X}$ is a quasi-field. If, for any $a \in A$, $h(a)$ denotes the set of all prime filters that contain a, then the authors show that the mapping $a \to h(a)$ is an isomorphic mapping of A into the quasi-field of all subsets of $\mathfrak{X}$ and thus A is isomorphic to a quasi-field of sets. *Ph. Dwinger* (Lafayette, Ind.).

Ellis, David. Remarks on Boolean functions. II. J. Math. Soc. Japan **8** (1956), 363–368.

The paper continues Part I [same J. **5** (1953), 345–350; MR **16**, 788]. It contains the statements and proofs of five lemmas and four theorems. The item with the longest proof (14 lines) is Lemma 3: A Boolean function $f(x, y) = \alpha xy + \beta x + \gamma y + \delta$ yields a quasigroup if and only if $\alpha = 0$ and $\beta = \gamma = 1$. Theorem 3 is a 23-line essay (requiring 9 lines of proof) on the semi-lattices arising in a Boolean algebra from Boolean functions $f(x, y)$, namely from the one-parameter family $xy + \lambda(x+y)$. *R. H. Bruck.*

Aubert, Karl Egil. A generalization of the ideal theory of commutative rings without finiteness assumptions. Math. Scand. **4** (1956), 209–230.

By axiomatizing the multiplication and subtraction of subsets of a commutative associative ring, the author generalizes some of the ideal theory for such rings to a corresponding theory for complete Boolean algebras with multiplication and subtraction operations. Such an algebra satisfying certain six axioms is called a d-algebra. Though a d-algebra in general may have no atoms, the author needs some remnant of the ring elements of his motivating example. This appears in Axiom III: Denoting by $\emptyset$ the zero element of the Boolean algebra, if $ab \cap c \neq \emptyset$ then there exist a_1, b_1 such that $\emptyset \neq a_1 \subset a$, $\emptyset \neq b_1 \subset b$, and $a_1 b_1 \subset c$. It is possible to extend the definitions of ideal, prime ideal, weak primary ideal and radical of an ideal to d-algebras and arrive at the result that if a is any ideal (element of a d-algebra) and p is a minimal prime ideal containing a (i.e., $a \subset p$) there corresponds a unique weak primary ideal q containing a and having radical p.

D. Zelinsky (Evanston, Ill.).

See also: Foundations, Theory of Sets, Logic: Wright. **Groups and Generalizations:** Benado; Banaschewski; Szász and Szendrei. **Topological Algebraic Structures:** Wallace. **General Topology:** Švarc.

Fields, Rings

Kawada, Yukiyosi; and Satake, Ichiro. Class formations. II. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1956), 353–389.

Soient k_0 un corps, Ω une extension algébrique galoisienne (donc séparable) de degré infini de k_0, $\mathfrak{K}$ l'ensemble des extensions de degré fini $k \subset \Omega$ de k_0, $\Omega^{(a)}(k)/k$ la sous-extension-abélienne maximale de Ω/k (où $k \in \mathfrak{K}$), $G_{k/k}$ le groupe de Galois d'une extension galoisienne K/k, k_N/k_0 la plus petite sous-extension de k/k_0 ($k \in \mathfrak{K}$) telle que k/k_N soit galoisienne. Si K'/k est une surextension galoisienne d'une extension galoisienne K/k ($k, K, K' \in \mathfrak{K}$), soit $\lambda_{K'/K}$ l'homomorphisme naturel de $G_{K'/k}$ sur $G_{K/k}$. Tout $G_{K/k}$-module A sera considéré également comme un $G_{K'/k}$-module, en posant, pour tout $\sigma' \in G_{K'/k}$ et pour tout $a \in A$, $\sigma' \cdot a = (\lambda_{K'/K} \cdot \sigma') \cdot a$. Si G est un groupe et si A est un G-module, l'ensemble des $a \in A$ préservés par les $\sigma \in G$ sera noté A^G, et le groupe des traces $\mathrm{Tr}_G a = \sum_{\sigma \in G} \sigma \cdot a$ des $a \in A$ sera noté $\mathrm{Tr}_G A$. On notera $H^{(n)}(G; A)$ (où n parcourt les entiers rationnels) le n-ième groupe de cohomologie de G par rapport à A. Soit Z le groupe additif des entiers rationnels. Il sera considéré, quelque soit le groupe G, comme un G-module, où G opère trivialement.

Soit $E(k)$ une fonction définie sur $\mathfrak{K}$ et telle que, pour tout $k \in \mathfrak{K}$, $E(k)$ soit un G_{k/k_N}-module; et soit, pour tous $k, K \in \mathfrak{K}$ tels que $k \subset K$, $\varphi_{k/K}$ un isomorphisme additif de $E(k)$ dans $E(K)$. Le système $(E(k); \varphi_{k/K})$ [$k \in \mathfrak{K}$; $k \subset K \in \mathfrak{K}$]

s'appelle une formation de classes ("class formation" en anglais) s'il satisfait aux conditions suivantes: I. Si $k \subset l \subset K$ $(k, l, K \in \mathfrak{K})$, on a $\varphi_{k/K} = \varphi_{l/K}\varphi_{k/l}$; II. Si K/k $(k, K \in \mathfrak{K})$ est galoisienne, on a $\varphi_{k/K} \cdot E(k) = E(K)^{G_{K/k}}$; III. Si $k \subset L \subset K$ $(k, L, K \in \mathfrak{K})$ et si L/k et K/k sont galoisiennes, on a, pour tout $\sigma \in G_{K/k}$, $\sigma\varphi_{L/K} = \varphi_{L/K}$ $(\lambda_{K/L} \cdot \sigma)$; IV. Si K/k est galoisienne $(k, K \in \mathfrak{K})$, on a, pour tout $n \in Z$, $H^{(n)}(G_{K/k}; E(K)) \cong H^{(n-2)}(G_{K/k}; Z)$ (ce qui, en vertu du théorème de Tate, équivaut, quand il s'agit de la totalité des extensions K/k de cette sorte, à $H^{(1)}(G_{K/k}; E(K)) = 0$ et $H^{(2)}(G_{K/k}; E(K)) \cong Z/nZ$).

Dans la partie I du travail considéré [Duke Math. J. 22 (1955), 165–177; MR 16, 907] il a été montré qu'une telle formation de classes permet, moyennant un choix cohérent, quoique non-canonique en général, des générateurs $f_{K/k}$ des $H^{(2)}(G_{K/k}; E(K))$ (où K/k parcourt les extensions galoisiennes telles que k, $K \in \mathfrak{K}$), de définir un symbole analogue, par ses propriétés formelles, à ceux d'Artin et de Hasse, quoique ne donnant pas lieu, en général, à une loi de réciprocité de type arithmétique. Ce symbole permet de représenter $G_{\Omega^{(\omega)}(k)/k}$ comme un certain groupe-quotient de $E(k)$ organisé par une certaine topologie, mais la description explicite de ce groupe topologique (et du treillis de ses sous-groupes fermés) reste un problème à résoudre dans chaque cas particulier. Dans la 1-ère partie citée du travail et dans la présente 2-me partie, les auteurs montrent que la plupart des théories connues décrivant la structure du treillis des extensions abéliennes (générales ou spéciales) de certaines catégories des corps (aussi bien les théories algébriques formelles, comme celle des extensions kummériennes d'un corps de caractéristique 0 contenant toutes les racines de l'unité ou celle des p-extensions abéliennes séparables des corps de caractéristique p, que les "théories des corps des classes") se formulent, d'un point de vue unique, à l'aide de la théorie des formations de classes.

La présente partie II du travail est consacrée aux divers cas, qui se présentent quand la caractéristique de k_0 est $p \neq 0$ et Ω est une p-extension infinie de k_0 (c'est-à-dire, un composé d'extensions finies de degrés puissances de p). Les auteurs considèrent, d'abord, le cas où k_0 est un corps quelconque de caractéristique $p \neq 0$ et où Ω en est la p-extension séparable maximale. Si $k \in \mathfrak{K}$, soient $\mathfrak{B}(k)$ l'anneau des vecteurs de Witt [J. Reine Angew. Math. 176 (1936), 126–140] $\xi = (x_0, x_1, \cdots, x_n, \cdots)$ $(x_0, x_1, \cdots, x_n, \cdots \in k)$ sur k, $\xi^p = (x_0^p, x_1^p, \cdots, x_n^p, \cdots)$, ρ l'opérateur linéaire tel que $\rho \cdot \xi = \xi^p - \xi$, Q^* le groupe additif des nombres rationnels de dénominateur puissance de p, $\mathfrak{W}(k) = (\mathfrak{B}(k)/\rho \cdot \mathfrak{B}(k)) \otimes (Q^*/Z)$ (où $\otimes$ est le signe de produit tensoriel). $\mathfrak{W}(k)$ est, d'une manière évidente, un G_{k/k_0}-module. Les auteurs prouvent que le groupe $(G_{\Omega^{(\omega)}(k)/k})^\wedge$ du groupe compact $G_{\Omega^{(\omega)}(k)/k}$ (c'est donc un groupe discret, admettant, d'une manière évidente, G_{k/k_0} comme groupe d'opérateurs) est G_{k/k_0}-isomorphe à $\mathfrak{W}(k)$, d'où résulte, en vertu du théorème de dualité de Pontrjagin, que $G_{\Omega^{(\omega)}(k)/k}$ est isomorphe au groupe compact $\mathfrak{W}(k)^\wedge$ des caractères du groupe $\mathfrak{W}(k)$. Si $k \subset K$, la trace $\mathrm{Tr}_{K/k}$ est un homomorphisme de $\mathfrak{B}(K)$ dans $\mathfrak{B}(k)$ qui commute avec ρ, d'où résulte que $\mathrm{Tr}_{K/k} = \mathrm{Tr}_{K/k} \otimes 1$ est un homomorphisme de $\mathfrak{W}(K)$ dans $\mathfrak{W}(k)$ et que son application duale $\mathrm{Cotr}_{k/K}$ applique continument $\mathfrak{W}(k)^\wedge$ dans $\mathfrak{W}(K)^\wedge$. On montre que le système $(E(k) = \mathfrak{W}(k)^\wedge, \varphi_{k/K} = \mathrm{Cotr}_{k/K})$ est une formation de classes et qu'il est possible de définir explicitement et canoniquement un isomorphisme de $G_{\Omega^{(\omega)}(k)/k}$ sur $E(k)$ et le "symbole d'Artin" correspondant. Dans le cas présent, les sous-groupes ouverts d'indice fini de $E(k) = \mathfrak{W}(k)^\wedge$ coïncident avec les groupes des annulateurs des sous-groupes d'ordre fini de $\mathfrak{W}(k)$.

Ensuite, deux cas particuliers de la théorie précédente sont considérées: celui où k_0 est le corps des séries des puissances sur un champ de Galois, et celui où il est un corps des fonctions algébriques d'une variable avec un corps des coefficients fini. Ces cas correspondent aux analogues, en caractéristique p, des théories locale et globale des corps de classes. Comme on doit s'y attendre on a, dans ces cas respectifs, les G_{k/k_0}-isomorphismes topologiques $\mathfrak{W}(k)^\wedge \cong k^*$ et $\mathfrak{W}(k)^\wedge \cong J_k/P_k$, où k^* est le groupe multiplicatif de k, et où J_k/P_k est le groupe des classes des idèles de k, quotient du groupe J_k de ses idèles par celui P_k de ses idèles principaux. Ces isomorphismes sont définis explicitement, en se servant, dans le premier cas, des propiétés du symbole $\mathrm{Res}^{(n)}\left(a^{(n)} \frac{db}{b}\right)$ de Witt (où $a^{(n)}$ est un vecteur de Witt de longueur n et où $b = b(t)$ est une série des puissances $\in k$), et, dans le second cas, de la théorie locale, qui vient d'être établie, et des calculs, assez analogues à ceux qui servent à établir, dans la théorie globale des corps de classes, la 2-me inégalité fondamentale. Il se trouve que ces isomorphismes des $\mathfrak{W}(k)^\wedge$ aves les k^* ou J_k/P_k correspondants transforment $\varphi_{k/K}$ en injections naturelles $k^* \to K^*$ et $J_k/P_k \to J_K/P_K$. Ainsi, $(E(k) = k^*$, $\varphi_{k/K}$ = injection naturelle) et $(E(k) = J_k/P_k$, $\varphi_{k/K}$ = injection naturelle) sont les formations de classes, qui permettent de fonder les théories des corps de classes considérées si l'on se limite aux p-extensions.

Les auteurs finissent en considérant le cas, où k_0 est un corps des fonctions algébriques avec un corps des constantes algébriquement clos de caractéristique p, et où Ω est la p-extension séparable non-ramifié maximale de k_0. Dans ce cas, si $k \in \mathfrak{K}$, $\Omega^{(\omega)}(k)$ est, puisque k/k_0 est non-ramifiée, l'extension abélienne séparable non-ramifiée maximale de k. Un vecteur de Witt $\xi = (x_0, x_1, \cdots, x_n, \cdots) \in \mathfrak{B}(k)$ est dit non-ramifié si, pour toute place $\hat{\mathfrak{p}}$ de la fermeture algébrique $\hat{k}$ de k, le vecteur $\rho^{-1} \cdot \xi$ (c'est-à-dire une solution η dans $\hat{k}$ de $\rho \cdot \eta = \xi$), considéré comme un vecteur $\in \mathfrak{B}$ $(\hat{k}_{\hat{\mathfrak{p}}})$ (où $\hat{k}_{\hat{\mathfrak{p}}}$ est le complété de $\hat{k}$ par rapport à $\hat{\mathfrak{p}}$) appartient à $\mathfrak{B}(k_{\mathfrak{p}})$, où $\mathfrak{p}$ est la place de k au-dessous de $\hat{\mathfrak{p}}$. Soit ${}^*\mathfrak{B}(k)$ le groupe additif des vecteurs non-ramifiés $\in \mathfrak{B}(k)$. On a $\rho \cdot \mathfrak{B}(k) \subseteq {}^*\mathfrak{B}(k)$ (car $\rho^{-1} \cdot (\rho \cdot \mathfrak{B}(k)) = \mathfrak{B}(k) \subseteq \mathfrak{B}(k_{\mathfrak{p}})$) et on montre que, si $k \subset K \subset \Omega$, $\mathrm{Tr}_{K/k}\, {}^*\mathfrak{B}(K) \subseteq {}^*\mathfrak{B}(k)$. Soit ${}^*\mathfrak{W}(k) = ({}^*\mathfrak{B}(k)/\rho \cdot \mathfrak{B}(k)) \otimes (Q^*/Z)$. Les auteurs montrent, en partant des résultats de Witt et de Safarevič sur les extensions non-ramifiées de tels corps k (tous les autres raisonnements étant analogues à ceux du premier cas), que $(G_{\Omega^{(\omega)}(k)/k})^\wedge \cong {}^*\mathfrak{W}(k)$, d'où résulte que $(E(k) = {}^*\mathfrak{W}(k)$, $\varphi_{k/K} = \mathrm{Cotr}_{k/K})$ est une formation de classes pour le cas considéré.

Les détails des démonstrations sont trop compliquées pour qu'on puisse les indiquer ici. *M. Krasner.*

Faith, Carl C. Extensions of normal bases and completely basic fields. Trans. Amer. Math. Soc. 85 (1957), 406–427.

Let $\mathfrak{R}/\mathfrak{F}$ be a normal separable field extension of finite degree n with Galois group $\mathfrak{G}$. It is well known that there exist elements $\omega \in \mathfrak{R}$ such that ω^S $(S \in \mathfrak{G})$ is a basis for $\mathfrak{R}/\mathfrak{F}$, and then ω is called a normal basis element for $\mathfrak{R}/\mathfrak{F}$. The author calls ω completely basic if it is basic for $\mathfrak{R}/\Delta$, where Δ runs through all fields between $\mathfrak{R}$ and $\mathfrak{F}$. The author gives examples of extensions $\mathfrak{R}/\mathfrak{F}$ for which no completely basic elements exist: but he shows that completely basic elements exist whenever $\mathfrak{F}$ has infinitely many elements and in some cases when it has a finite

number of elements; for example, when the degree of the extension is a power of the characteristic. The author calls the extension $\mathfrak{R}/\mathfrak{F}$ completely basic if every normal basis element is completely basic. He shows that Kummer extensions are completely basic and gives a necessary and sufficient criterion to decide whether a cyclic extension $\mathfrak{R}/\mathfrak{F}$ of prime power degree $n=p^e$ is completely basic provided that $\mathfrak{F}$ contains the pth roots of unity. The criterion depends on the roots of unity contained in $\mathfrak{R}$ and $\mathfrak{F}$ whose order is a power of p. The referee considers the author's historical remarks as probably misleading. Deuring's proof of the finite basis theorem [Math. Ann. **107** (1932), 140–144] purports to cover all cases, so if the author gives the reference stating that it covers only the case when the degree is not divisible by the characteristic he is under a duty, at least in the reviewer's opinion, to warn the reader where he believes the fallacy to lie in the remaining case. The author's Lemma 1.1 "which we believe is new" is substantially identical with Lemma 3 of Cassels and Wall [J. London Math. Soc. **25** (1950), 259–264, item 5 of the author's bibliography; MR **12**, 237] who state it to be similar to a criterion of Hensel [J. Reine Angew. Math. **103** (1888), 230–237]. *J. W. S. Cassels.*

Tominaga, Hisao. Galois theory of simple rings. II. Math. J. Okayama Univ. **6** (1957), 153–170.

Comme dans la première partie de son travail [même J. **6** (1956), 29–48; MR **18**, 636] l'auteur s'occupe d'étendre la théorie de Galois pour deux anneaux simples $R \supset S$ lorsqu'on ne suppose pas $[R:S]$ fini. Il montre qu'on peut améliorer quelque peu les restrictions de finitude de la 1-re partie; ici, les principales hypothèses sont: (1) pour toute partie finie F de R, l'anneau $S(F)$ engendré par $S \cup F$ est un module à gauche de dimension finie sur S; (2) $\mathfrak{G}$ étant un groupe de S-automorphismes de R, pour toute partie finie F de R, le sous-anneau de R engendré par S et les transformés de F par $\mathfrak{G}$ est un S-module à gauche de dimension finie. L'auteur étudie les relations entre ces hypothèses, celles de la 1-re partie et diverses propriétés de $\mathfrak{G}$ (compacité ou compacité locale); et leurs conséquences pour les théorèmes de la théorie de Galois; il n'est pas possible de résumer ici les nombreux énoncés du mémoire. *J. Dieudonné* (Evanston, Ill.).

Nagahara, Takasi. On generating elements of Galois extensions of division rings. Math. J. Okayama Univ. **6** (1957), 181–190.

It is a result of A. A. Albert [Bull. Amer. Math. Soc. **50** (1944), 786–788; MR **6**, 115] that every finite-dimensional division algebra has a two-element generation. The problem of the generation of division rings has been attacked with renewed vigor since the announcement of the Galois theory for such rings by Jacobson [Amer. J. Math. **69** (1947), 27–36; MR **9**, 4] and Cartan [C. R. Acad. Sci. Paris **224** (1947), 87–89, 249–251; MR **9**, 5]. In fact, Cartan's Théorème 4 of Ann. Sci. Ecole Norm. Sup. (3) **64** (1947), 59–77 [MR **9**, 325] implies that a non-commutative finite-dimensional division algebra is generated by the conjugates of any non-central element, a result extended to arbitrary division algebras by Brauer [Bull. Amer. Math. Soc. **55** (1949), 619–620; MR **10**, 676] and Hua [Proc. Acad. Sci. U.S.A. **35** (1949), 533–537; MR **11**, 155]. Kasch [[1] J. Reine Angew. Math. **189** (1951), 150–159; MR **14**, 239] obtained that the elements of the Albert generation can be taken to be conjugates as a corollary to the following result in [1]: (A) If K/L is finite-dimensional and Galois, then K/L is generated by two conjugates, $K=L(u, d^{-1}ud)$, provided (*) the center of the centralizer L' of L in K is separable over the center C of K. Later [[2] Math. Ann. **126** (1953), 447–463; MR **15**, 597] Kasch proved (A) without the assumption (*). Nagahara [[3] Math. J. Okayama Univ. **6** (1956), 23–28; MR **19**, 9] has considered the generation of certain intermediate division subrings of the Galois extension K/L: (B) If $K \supseteq D \supseteq L$, and if $x^{-1}Dx=D$ for all nonzero $x \in L'$, then $D=L(u, d^{-1}ud)$ provided (*) holds. Now in the present article the author employs Kasch's later result [2], and other results of [1–3], to establish (B) without assuming (*). The strongly computational techniques of the earlier article [3] are continued in the present article. However there is much that is new {and interesting once the reader penetrates the austerity of the style of writing, and the occasionally cumbersome notation}. Moreover (B) is not necessarily the most striking result; viz., if L is a commutative field not contained in the center C of K, then K/L is simply generated: $K=L(d)$ [Lemma 7]. A consequence of this result (not drawn by the author) is, in the two element Albert generation $A=F(x,y)$ of a central division algebra A over F, that as (x, y) ranges over all two element generators, x ranges over all non-central elements of A, i.e., given $x \notin F$ there exists a y such that $A=F(x, y)$.

{Reviewer's comments. (1) In the third line of the proof of Lemma 5 the author refers to Nobusawa, Osaka Math. J. **8** (1956), 43–50 [MR **18**, 7], presumably to the "Theorem" on page 49. In view of the objections raised by Dieudonné [MR **18**, 7] this "Theorem" cannot be regarded as proved. In this particular instance the difficulties are obviated by the fact that the author simultaneously appeals to the valid Corollary 2 of [3]. (2) There are trivial misprints involving subscripts to the z_i on lines 4 and 9 of page 184. (3) I have found no allusion in the text to the article listed in the References as [6].} *C. C. Faith.*

Grätzer, G.; und Schmidt, E. T. Über die Anordnung von Ringen. Acta Math. Acad. Sci. Hungar. **8** (1957), 259–260.

Let R be a ring with no zero divisors, and let I be a non-zero ideal of R. Suppose that I is a linearly ordered ring. Define $x \in R$ positive if there exists a positive element $a \in I$ such that ax is a positive element in I. This is the unique way to extend the order of I to an ordering of R. *P. F. Conrad* (New Orleans, La.).

San Soucie, R. L. On the left nucleus of a Bruck ring. Proc. Amer. Math. Soc. **7** (1956), 961–962.

This note gives a partial answer to the following question: Is every finite right alternative division ring a field? All finite alternative division rings are known to be fields and all right alternative division rings of characteristic other then 2 are known to be alternative. Here the author considers the only non-empty class of right alternative (but not alternative) division rings now available in the literature, the so-called Bruck rings [San Soucie, Amer. J. Math. **77** (1955), 190–196; MR **16**, 896, 1337]. He shows that if B is a Bruck ring with left nucleus F, then F contains a non-square. Since F is a field of characteristic 2, it follows that F and B are infinite. *R. H. Bruck* (Madison, Wis.).

Stöcker, Claus. Alternative Divisionsringe beliebiger Charakteristik. Math. Ann. **132** (1956), 17–42.

Let there be given elements x, y, z of an alternative division ring R such that $(xy)z \neq x(yz)$. The author shows how to construct elements u, v of R such that $1+2u \neq 0$

and $uv+vu=v\neq 0$. He then shows the existence (but the proof is non-constructive at one point) of a third element $w\neq 0$ such that $uw+wu=w$, $vw+wv=0$ and $w(vu)+(wv)u=wv$. And finally he proves that the eight elements $1, u, v, vu, w, wu, wv, w(vu)$ form a basis of R over its centre. In this way he gives a new proof (independent of characteristic) of the known theorem that R is a Cayley division algebra. The main feature of the paper is its elementary and constructive approach; as the author makes clear, many of the tools and ideas are due to others.

R. H. Bruck (Madison, Wis.).

Kleinfeld, Erwin. Alternative nil rings. Ann. of Math. (2) **66** (1957), 395–399.

The author has previously proved that a simple, alternative ring is either associative, or a Cayley-Dickson algebra or a nil ring. [Ann. of Math. (2) **58** (1953), 544–547; MR **15**, 392]. The present paper pleasantly reduces the size of limbo by proving that simple, alternative nil rings of characteristic not 3 are associative. This is an immediate corollary of the more general theorem that prime, alternative nil rings of characteristic not 3 are associative. Both of the results mentioned are speedily deduced from the following theorem, whose proof takes up most of the paper and which is not without some interest in itself. Let R be an alternative ring of characteristic not 3 without non-zero ideals A such that $A^2=0$. Then either $R=0$ or the nucleus of R contains a non-zero element. Since the author has previously shown that alternative, non-associative rings without proper two-sided ideals are also without proper one-sided ideals [Amer. J. Math. **77** (1955), 725–730; MR **17**, 231], simple, alternative, non-associative nil rings of characteristic 3 must be very unpleasant, if, indeed, they exist at all.

R. L. San Soucie (Buffalo, N.Y.).

Thierrin, Gabriel. Sur les idéaux complètement premiers d'un anneau quelconque. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 124–132.

Let A be an arbitrary associative ring. An ideal M of A is called completely prime if and only if $ab\in M$ implies $a\in M$ or $b\in M$, and called compressed provided, for any positive integer n, $a_1^2a_2^2\cdots a_n^2\in M$ implies $a_1a_2\cdots a_n\in M$. A ring with compressed zero ideal is called a compressed ring. An element a of A is called metanilpotent if and only if there exist positive integers $r_1, r_2, \cdots, r_n$ and elements $a_1, a_2, \cdots, a_n$ in A such that $a=a_1a_2\cdots a_n$ and $a_1^{r_1}a_2^{r_2}\cdots a_n^{r_n}=0$. Let M be an ideal of A and let $R_1^*(M)$ be the set of elements $x\in A$ such that there exists a decomposition $x=x_1x_2\cdots x_n$ with $x_1^2x_2^2\cdots x_n^2\in M$ for some n. Let $R_1(M)$ be the ideal generated by $R_1^*(M)$ and, in general, define $R_k(M)=R_1[R_{k-1}(M)]$ $(k=2, 3, \cdots)$. Clearly the $R_k(M)$ are ideals. Set $R(M)=\bigcup_{k=1}^{\infty}R_k(M)$ and call $R(M)$ the compressed radical of the ideal M. Finally, a completely prime (prime) ideal P of A is called a minimal completely prime (prime) ideal belonging to M if and only if $M\subseteq P$ and there does not exist a completely prime (prime) ideal P' such that $M\subseteq P'\subset P$, $P'\neq P$. Typical results: If M is an ideal of A, A/M is compressed if and only if M is a compressed ideal. A necessary and sufficient condition for a ring A to be compressed is that it contain no metanilpotent elements different from zero. The compressed radical $R(M)$ of an ideal M is a compressed ideal, contains the McCoy radical of M [Amer. J. Math. **71** (1949), 823–833; MR **11**, 311], is the intersection of all compressed ideals containing M, and is the intersection of all the minimal completely prime ideals belonging to M. The two main theorems are: For $A\neq 0$ to be isomorphic to a subdirect sum of rings without divisors of zero it is both necessary and sufficient that A be compressed. Every subdirectly irreducible, non-zero, compressed ring is a ring without divisors of zero. {Reviewer's remark: According to one of the results listed above, the last theorem may be paraphrased to read: Every non-zero, subdirectly irreducible ring without metanilpotent elements is a ring without divisors of zero. We note that "nilpotent" may be substituted for "metanilpotent" in the last sentence without affecting the validity of the theorem and that the proof of the remark is independent of the theory developed in the paper.} *R. L. San Soucie.*

Yoshida, Michio. On polynomial extensions of rings. Canad. J. Math. **8** (1956), 1–2.

Two new proofs are given for the theorem of Butts, Hall and Mann [Canad. J. Math. **6** (1954), 471–473; MR **17**, 349] on integral closure of the ring of polynomials over an integrally closed integral domain with unit element. In addition, it is proved that the ring of polynomials over a completely integrally closed integral domain is also a completely integrally closed integral domain.

É. G. Šul'geĭfer (RŽMat **1957**, no. 2939).

Schreiber, M. Compactness of the structure space of a ring. Proc. Amer. Math. Soc. **8** (1957), 684–685.

For any ring A, let S_A denote the structure space of A in the sense of Jacobson [cf. Structure of rings, Amer. Math. Soc. Colloq. Publ., vol. 37, Providence, R.I., 1956; MR **18**, 373]. The author observes that if every two-sided ideal of A is finitely generated, then S_A is compact. Conversely, if S_A is compact, then A contains a finitely generated two-sided ideal I such that S_A and S_I are homeomorphic.

M. Henriksen (Lafayette, Ind.).

Rees, D. A note on general ideals. J. London Math. Soc. **32** (1957), 181–186.

A general ideal of rank k is a proper ideal $\mathfrak{g}_k$ of a noetherian ring A which has a set of generators $g_1, \cdots, g_k$ such that $(g_1, \cdots, g_{t-1}):g_t=(g_1, \cdots, g_{t-1})$ [see D. Rees, Proc. Cambridge Philos. Soc. **52** (1956), 605–610; MR **18**, 277]. The grade of an ideal $\mathfrak{a}$ of A is defined to be that integer k such that $\mathfrak{a}$ contains a general ideal of rank k but no such ideal of rank $k+1$. The main theorem of this note is that if $\mathfrak{p}$ is a prime ideal of A of grade k, and if $\mathfrak{g}_k$ denotes a general ideal of rank k contained in $\mathfrak{p}$, then $\mu_{\mathfrak{p}}(\mathfrak{g}_k)$ depends only on $\mathfrak{p}$, where $\mu_{\mathfrak{p}}(\mathfrak{g}_k)$ is the number of irreducible ideals belonging to $\mathfrak{p}$ in an irredundant decomposition of $\mathfrak{g}_k$ as the intersection of irreducible ideals.

The author first reduces the problem to the case where A is a local ring and $\mathfrak{p}$ is the maximal ideal. The following two theorems are then employed to conclude the problem. Theorem A. Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{g}_k$, $\mathfrak{g}_k'$ be two general ideals of rank k contained in $\mathfrak{a}$. Then the two A-modules $(\mathfrak{g}_k:\mathfrak{a})/\mathfrak{g}_k$ and $(\mathfrak{g}_k':\mathfrak{a})/\mathfrak{g}_k'$ are isomorphic [loc. cit.]. Theorem B. If $\mathfrak{a}$ is an ideal of a local ring Q with maximal ideal $\mathfrak{m}$, then $\mu_{\mathfrak{m}}(\mathfrak{a})$ is equal to the length of the module $\mathfrak{a}:\mathfrak{m}/\mathfrak{a}$. The author includes a proof of this last result.

D. Buchsbaum (Providence, R.I.).

McCoy, N. H. Certain classes of ideals in polynomial rings. Canad. J. Math. **9** (1957), 352–362.

Let σ be a property of ideals (right ideals) of a ring and let an ideal with this property be called a σ-ideal. For an ideal A of a ring, let A_σ be the intersection of all σ-ideals containing A. If for each σ-ideal A of R, $A[x]$ is a σ-ideal

of $R[x]$, and for each σ-ideal A of $R[x]$, $A \cap R$ is a σ-ideal of R, then σ will be called a (*)-property. If σ means "prime" or "semiprime" (for ideals or right ideals), then σ is shown to be a (*)-property. Theorem: If R is a σ-ideal, if for every ideal A and every σ-ideal B of $R[x]$ it is true that $AR[x] \subset B$ implies $A \subset B$, and if σ is a (*)-property, then $(A[x])_\sigma = A_\sigma[x]$ for each ideal A of R. A corollary of this theorem is that $K[x]$ is the prime radical of $R[x]$ if K is the prime radical of R. There are other similar results relating the ideal structures of R and $R[x]$. For example, if R is semi-prime, the annihilating ideals of $R[x]$ are of the form $A[x]$, A an annihilating ideal of R. An example is given to show that there may be prime right ideals in $R[x]$ other than those of the form $A[x]$, A a prime right ideal of R. *R. E. Johnson* (Northampton, Mass.).

Amitsur, S. A. A generalization of Hilbert's Nullstellensatz. Proc. Amer. Math. Soc. **8** (1957), 649–656.

Let F be a field, $\overline{F}$ its algebraic closure, and $F[x_1, \cdots, x_n] = F[x]$ the free algebra with a unit generated by the noncommutative indeterminates $x_1, \cdots, x_n$ over F. Let $\overline{F}_n$ be the algebra of all square matrices of order n over $\overline{F}$ and $\mathfrak{M}_n$ the ideal of all polynomials $f(x) \in F[x]$ such that $f(x_1, \cdots, x_n) = 0$ in $\overline{F}_n$. Let $I(G)$ be the ideal generated by G, where G is a set of polynomials belonging to $F[x]$. Let $Z_k(G)$, $Z_k^R(G)$ and $Z_\infty(G)$ be the sets of polynomials $f(x) \in F[x]$ such that $f(x)$ vanishes for all zeros of G which lie in $\overline{F}_k$, $f(x)$ vanishes for all regular zeros of G in $\overline{F}_r$ ($r \leqq k$), and $f(x)$ vanishes for all zeros of G which lie in primitive rings, respectively. Here a regular zero is a set of elements $(d_1, \cdots, d_n)$ belonging to $\overline{F}_r$, which is a zero of the polynomials of G, such that if Z_r is the centre of $\overline{F}_r$ then $\overline{F}_r = Z_r(d_1, \cdots, d_n)$. Let $Q_k = (I(G), \mathfrak{M}_k)$ and denote by J_k/Q_k the Jacobson radical of $F[x]/Q_k$. Then the author proves the following: i) $Z_k(G) \subseteq J_k$; ii) $Z_k^R(G) = J_k$; iii) the Jacobson radical of a finitely generated algebra which satisfies an identity is a nil ideal; and the following generalizations of the Hilbert Nullstellensatz: iv) if $f \in Z_k(G)$, $f^m(x) \in Q_k$ for some m; and v) if $f \in Z_\infty(G)$, $f^m(x) \in I(G)$ for some m. *P. A. Abellanas* (Madrid).

Goldman, Lawrence. Specialization and Picard-Vessiot theory. Trans. Amer. Math. Soc. **85** (1957), 327–356.

The author is interested in what happens to the Galois group of a homogeneous linear ordinary differential equation when the coefficients are specialized. The first half of the paper is devoted to fairly general theorems. As their statement is rather lengthy, we describe the first two only; these give an idea of the nature of the investigation.

Let F be an ordinary differential field of characteristic 0 with algebraically closed field of constants C. If F_1 is any extension of F and β is any constant transcendental over F_1, the field $F_1((\beta))$ of formal power series in β over F_1 has an obvious differential field structure. If $\bar{t} = t + \sum_{i \geq 1} t_i \beta^i \in F_1((\beta))$, obviously $\bar{t}$ is a specialization (in the sense of differential algebra) over F of $\bar{t}$, as well as of any element t of which $\bar{t}$ is a generic specialization. The author in this case calls $\bar{t}$ an "analytic" specialization of t over F. This definition extends in an obvious way to an arbitrary family of elements $(t_i)_{i \in I}$. It turns out that most specializations (in a certain precise sense) are analytic. Now let $L(t, y) = \sum_{0 \leq i \leq n} a_i(t) y^{(n-i)} = 0$ be a differential equation with coefficients in $F\{t\} = F\{t_1, \cdots, t_r\}$. If $\bar{t}$ is an analytic specialization of t over F such that the field of constants of $F(\bar{t})$ is C and if $\bar{\pi} = (\bar{\pi}_1, \cdots, \bar{\pi}_n)$ is a fundamental system of solutions of $L(\bar{t}, y) = 0$, then (Theorem 1) there exists a fundamental system π of solutions of $L(t, y) = 0$ such that $(\bar{\pi}, \bar{t})$ is an analytic specialization of (π, t) over F. Furthermore (Theorem 2), if the coefficients $f = (f_1, f_2, \cdots)$ of $\bar{t}$ have the property that the field of constants of $F\langle \bar{t}, f, \bar{\pi} \rangle$ is C, then there exists an algebraically closed field of constants E containing the field of constants of $F\langle t, \pi \rangle$ such that the Galois group of $L(t, y)$ over $F\langle t, E \rangle$, which may be identified with an algebraic matric group over E, contains a subgroup K such that there exists a surjective homomorphism $b \to \bar{b}$ of K onto the Galois group H of $L(\bar{t}, y)$ over $F\langle \bar{t}, f \rangle$ for which $(\bar{t}, \bar{\pi}, (\bar{b})_{b \in K})$ is an analytic specialization of $(t, \pi, (b)_{b \in K})$ over F. As a corollary, if in addition the field of constants of $F\langle t, \pi \rangle$ is C, then H may be identified with a subgroup of the Galois group of $L(t, y)$ over $F\langle t \rangle$.

When $t = (t_1, \cdots, t_r)$ is a family of differential indeterminates all the above hypotheses can be secured. In the second half of the paper it is assumed that this is the case, and that each $a_i(\bar{t}) \in C\{\bar{t}\}$. Let the Galois group of $L(t, y)$ over $C\langle t \rangle$, identified with an algebraic matric group by means of a fundamental system of solution π, be denoted by G. The author calls $L(t, y) = 0$ a "generic equation with group G" provided the following three conditions are satisfied: (1) $C\langle t \rangle \subset C\langle \pi \rangle$; (2) for every specialization $(\bar{t}, \bar{\pi})$ of (t, π) over C such that the field of constants of $\langle \bar{t}, \bar{\pi} \rangle$ is C, $\bar{\pi}$ is linearly independent over C, and $a_0(\bar{t}) \neq 0$, the Galois group of $L(\bar{t}, y)$ over $C\langle \bar{t} \rangle$ is a subgroup of G; (3) for every differential field F with field of constants C and every differential equation $\sum a_i y^{(n-i)} = 0$ with coefficients in F, $a_0 = 1$, and Galois group a subgroup of G, there exists a specialization $\bar{t}$ of t over F such that $a_0(\bar{t}) \neq 0$ and $a_i = a_i(\bar{t})/a_0(\bar{t})$ $(1 \leqq i \leqq n)$. It is easy to see that r must equal n. The second half of the paper is devoted mainly to obtaining a generic equation with each of the classical groups. *E. R. Kolchin* (New York, N.Y.).

See also: Partial Order, Lattices: Aubert. **Groups and Generalizations:** Landin and Reiner. **Homological Algebra:** Auslander; Higman. **Theory of Algebraic Numbers:** Albert; Krasner; Nakano (two articles); Lamprecht. **Algebraic Geometry:** Nagata.

Algebras

Gurevič, G. B. Linear Lie algebras of dimension n^2-1 or n^2-2. Tul'sk. Meh. Inst. Trudy. 1953, no. **6**, 3–7. (Russian)

In the Lie algebra of all complex matrices of order n all subalgebras of dimension n^2-1 or n^2-2 are determined. The totality of all matrices with vanishing trace forms a subalgebra of dimension n^2-1. Other subalgebras of dimension n^2-1 exist only for $n=2$. They are all conjugate to one another. Subalgebras of dimension n^2-2 exist only for $n=2$ and $n=3$ (in each case with two classes of conjugate subalgebras). In deducing these results the author uses concepts introduced by him earlier [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13** (1949), 403–416; MR **11**, 156]. *E. B. Dynkin* (RŽMat 1954, no. 3248).

Karpelevič, F. I. The simple subalgebras of the real Lie algebras. Trudy Moskov. Mat. Obšč. **4** (1955), 3–112. (Russian)

Let R be a real Lie algebra, $[R]$ its complex hull and $\mathfrak{L}(R)$ the class of real forms of the algebra $[R]$ which are conjugate to R. Let G be a subalgebra of R. We shall write $\mathfrak{L}(G) \subset \mathfrak{L}(R)$ if for every algebra $G' \in \mathfrak{L}(G)$ there

exists an algebra $R' \in \mathfrak{L}(R)$ such that $G' \subset R'$. Since the semi-simple complex subalgebras of simple complex Lie algebras have been precisely classified up to conjugacy, the problem of finding the simple subalgebras of real forms for the classical algebras can be naturally stated thus: given a semi-simple complex subalgebra $[G]$ of the algebra $[R]$, determine when $\mathfrak{L}(G) \subset \mathfrak{L}(R)$.

Let $\mathfrak{M}(G)$ be the class of involutive automorphisms of the algebra $[G]$ corresponding to the class $\mathfrak{L}(G)$ according to E. Cartan. Suppose $[G] \subset [R]$. We shall write $\mathfrak{M}(G) \subset \mathfrak{M}(R)$ if every automorphism of $[G]$ from the class $\mathfrak{M}(G)$ can be extended by some automorphism of $[R]$ from the class $\mathfrak{M}[R]$. It turns out that $\mathfrak{L}(G) \subset \mathfrak{L}(R)$ if and only if $\mathfrak{M}(G) \subset \mathfrak{M}(R)$. In this way the problem of classification of subalgebras is reduced to the problem of extension of involutive automorphisms of the algebra $[G]$ to all of algebra $[R]$.

If $[R]$ is a classical algebra, then the imbedding ϕ of $[G]$ in $[R]$ is a linear representation of $[G]$. In case the representation is irreducible and G is simple, the author deduces explicit formulas, which enable him, according to the highest weight of the representation ϕ, to find out how the involutive automorphisms of $[G]$ are extended to $[R]$. At the same time the problem is solved of classifying the irreducible simple subalgebras of real forms of $[R]$ up to conjugacy in $[R]$.

The author calls an automorphism of the algebra R quasi-inner if it can be extended to an inner automorphism of $[R]$. The subalgebras G and G' of R are called quasi-conjugate if they are mapped on each other by a quasi-inner automorphism of R. The problem arises of classifying the real subalgebras of R up to quasi-conjugacy. It turns out that if two irreducible semisimple subalgebras of a real form R of a classical algebra $[R]$ are conjugate in $[R]$, then they are quasi-conjugate in R. Thus a new statement of the problem of classifying the subalgebras is obtained from the old one only in the case that the representation ϕ: $[G] \to [R]$ is reducible. In this case, in order to characterize the subalgebra G up to quasi-conjugacy, it is not enough, in general, to give the highest weights of the irreducible components of ϕ, and certain other properties must also be given. As an example of such a property the author introduces a certain set of integers, which permits an effective classification of reducible subalgebras up to quasi-conjugacy.

It is remarked that these results allow us to solve a series of problems of the theory of linear representations; for example, the question whether, for a given representation there exists an invariant Hermitian form with given signature.

In a supplement a study is made of the structure of the group of quasi-inner automorphisms of various real forms of classical algebras.

A. L. Oniščik [RŽMat **1956**, no. 5759].

See also: Fields, Rings: Tominaga; Nagahara; Stöcker. **Homological Algebra:** Higman. **Lie Groups and Algebras:** Séminaire "Sophus Lie".

Groups and Generalizations

Benado, Mihail. Sur la théorie générale des produits réguliers. C. R. Acad. Sci. Paris **244** (1957), 1595–1597.

A set of axioms for a product operation on a complete lattice is given which abstracts the properties of subsystem products for groups, Lie algebras, and rings. The concept of regular products of groups introduced by O. N. Golovin [Mat. Sb. N.S. **27(69)** (1950), 427–454; MR **12**, 672] and studied by R. R. Struik [Trans. Amer. Math. Soc. **81** (1956), 425–452; MR **17**, 1051] is generalized.

R. P. Dilworth (Pasadena, Calif.).

Gravett, K. A. H. Valued linear spaces. Quart. J. Math., Oxford Ser. (2) **6** (1955), 309–315.

The main result of this paper is an embedding theorem for valued linear spaces. Hahn's theorem for ordered abelian groups (see the following review) is a special case of this theorem. (L, Δ, d) is a valued linear space if L is a linear space over a division ring K, Δ is a totally ordered set with minimal element μ, and d is a mapping of L onto Δ such that for $x, y \in L$: (1) $d(x)=\mu$ if and only if $x=0$; (2) $d(x)=d(kx)$ for all $0 \neq k \in K$; (3) $d(x+y) \leqq \max\{d(x), d(y)\}$. For each $\delta \in \Delta - \{\mu\}$, let $A(\delta)$ ($B(\delta)$) be the subspace of all $x \in L$ for which $d(x) \leqq \delta$ ($d(x) < \delta$). Let $A(\mu)=B(\mu)=0$, and let $C(\delta)=A(\delta)/B(\delta)$ for all $\delta \in \Delta$. Let W be the space of all mappings x of Δ into the join of the $C(\delta)$ for which $x(\delta) \in C(\delta)$ for all $\delta \in \Delta$, and such that $x(\delta)=0$ except for an inversely well-ordered subset of Δ. For each $x \in W$, define $d(x)$ to be the maximal element $\delta \in \Delta$ such that $x(\delta) \neq 0$. Then (W, Δ, d) is a valued linear space, and there exists a value-preserving isomorphism f of L into W. Moreover, W is a maximal immediate extension of the image of f. This theorem is a special case of a theorem by the reviewer [Amer. J. Math. **75** (1953), 1–29; MR **14**, 842], but the proof is entirely different. The author makes use of the standard valuation techniques. The concept of a pseudo-convergent sequence is central. L is pseudo-complete if and only if L has no proper immediate extension.

P. F. Conrad (New Orleans, La.).

Banaschewski, Bernhard. Totalgeordnete Moduln. Arch. Math. **7** (1957), 430–440.

This is another proof of Hahn's embedding theorem for ordered abelian groups. For other proofs see the above review or the reviewer's paper cited therein or a paper by Clifford [Proc. Amer. Math. Soc. **5** (1954), 860–863; MR **16**, 792]. This proof is more elementary than any of the above, and can be applied to prove the generalization obtained by Gravett. Let M be a vector space over the division ring F, and let $\mathscr{K}$ be a non-empty set of subspaces of M. Then there exists a function γ of $\mathscr{K}$ into the set of all subspaces of M such that for $H, K \in \mathscr{K}$: (1) $K \cap \gamma(K) = 0$ and $K \oplus \gamma(K) = M$; (2) if $K \subseteq H$, then $\gamma(K) \supseteq \gamma(H)$. Next, let M be an ordered abelian group that is also a vector space over the rationals. For each pair of convex subgroups K, K' of M such that K covers K', choose a subspace $\hat{K}$ of K so that $K = \hat{K} \oplus K'$. Then $\hat{K}$ is o-isomorphic to a subgroup of the reals. Let $\mathscr{K}$ be the set of all convex subgroups K of M that cover. Then by the above lemma there exists a function γ so that $M = \gamma(K) \oplus \hat{K} \oplus K'$. For each $x \in M$ let $\bar{x}(K)$ be the component of x in $\hat{K}$. $\bar{x}$ is essentially a real-valued function with domain $\mathscr{K}$. The mapping of $x \in M$ upon $\bar{x}$ is the desired embedding isomorphism. There is a unique extension of the ordering of an abelian group to an ordering of its rational hull (or completion). Thus the embedding theorem applies to arbitrary ordered abelian groups. The remaining theorems in this paper are fairly well known.

P. F. Conrad (New Orleans, La.).

Čarin, V. S. On groups possessing solvable increasing invariant series. Mat. Sb. N. S. **41(83)** (1957), 297–316. (Russian)

The author studies groups G possessing invariant series

(i.s.), i.e., series $1=G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset \cdots \subset G_\gamma = G$, where (i) all G_α are normal subgroups of G; (ii) if β is a limit ordinal then G_β is the union of all G_α with $\alpha<\beta$; (iii) all factors $G_{\alpha+1}/G_\alpha$ are abelian. The i.s. is called central if, for every α, $G_{\alpha+1}/G_\alpha$ is contained in the center of G/G_α. The following is proved. If every factor of the i.s. has finite rank, and if G is either locally nilpotent or has no proper subgroup of finite index, then G possesses a central i.s. If the ranks of all factors are smaller than a fixed natural number, then there exists a natural number s such that the sth derived group $G^{(s)}$ has a central i.s. If all factors have rank 1, then we may set $s=1$ (this is a generalization of the well-known case where G is finite). If every factor is generated by one or two elements and if G is periodic, then $G^{(3)}$ has a central i.s. The case where every factor is isomorphic with a subgroup of the additive group of the rationals is also considered.

Denote by $R(G)$ the maximal locally nilpotent normal subgroup of a group G [the uniqueness of $R(G)$ was first proved by Plotkin, Mat. Sb. N.S. **37(79)** (1955), 507–526; MR **17**, 709], and consider the series $1=R_0 \subset \cdots \subset R_\alpha \subset R_{\alpha+1} \subset \cdots$, where $R_{\alpha+1}/R_\alpha = R(G/R_\alpha)$. If $R_\gamma = G$ for some ordinal γ, then G is called a radical group. Theorem: Let G be a radical group. Let H be a normal subgroup possessing an i.s. consisting of normal subgroups of G and having factors of finite rank. If G satisfies the minimal condition for normal subgroups then H satisfies the minimal condition for subgroups. *R. Ree* (Vancouver, B.C.).

Gruenberg, K. W. Residual properties of infinite soluble groups. Proc. London Math. Soc. (3) **7** (1957), 29–62.

Let P be a property of groups. A group G is called residually-P if for every element $g\neq 1$ in G there exists a normal subgroup $K=K(g)$ not containing g and such that G/K has the property P. Let 'finite-p' be the property of being of order a finite power of the prime p. Let $X=X_1>X_2>X_3>\cdots$ be the lower central series of the free group X and, for $r>1$, let $X_{i_1,i_2,\ldots,i_r}$ be the i_r-th term of the lower central series of $X_{i_1,i_2,\ldots,i_{r-1}}$. It is known that X is residually finite-p for all primes p. The same is here proved to hold for the factor-groups $X/X_{i_1,i_2,\ldots,i_r}$ provided $p \geqq \max(i_1, \cdots, i_{r-1})$.

More general results concern the residual properties of free products. Let P be any property such that subgroups and finite direct products of P-groups are again P-groups. Suppose further that, whenever two factor groups G/H and H/K are P-groups, where K need not be normal in G, there always exists a subgroup L of K, normal in G and such that G/L is a P-group. Let F be a free product of residually-P groups, let F^* be the kernel of the natural homomorphism of F onto the corresponding direct product, and let $V(F^*)$ be any verbal subgroup of F^*, generated by the values assumed by any given set of words for arbitrary arguments in F^*. It is then shown that $F/V(F^*)$ will be a residually-P group, provided $X/V(X)$ is residually-P for every free group X. Corollary: if the free factors of F are all Abelian and residually finite-p, then $F/F^{(k)}$ is residually finite-p, $F^{(k)}$ being any term of the derived series of F. *P. Hall.*

Schenkman, Eugene. A characterization of some metacyclic groups. Proc. Amer. Math. Soc. **8** (1957), 664–667.

A metacyclic group for which the order of the commutator subgroup is prime to its index is characterized as a solvable group such that every member of a composition series is generated by some power of the group. This generalizes a result of F. Szasz [Acta Math. Acad. Sci. Hungar. **6** (1955), 475–477; MR **17**, 940]. *C. C. Faith.*

Adney, J. E. On the power of a prime dividing the order of a group of automorphisms. Proc. Amer. Math. Soc. **8** (1957), 627–633.

The following theorem is proved. (1) If G is a finite group with an abelian Sylow subgroup of order p^n, then p^{n-1} divides the order of the automorphism group $A(G)$. An example is given to show that (1) is false for non-abelian Sylow subgroups. The proof of (1) is accomplished by a reduction to: (2) if $H \supseteq K \neq 1$ are finite Abelian p-groups, then the group of automorphisms of H fixing H/K elementwise has order divisible by $o(K)/p$. However, there is a gap in the proof of (2), since the author does not prove the existence of a group of automorphisms of the right size and type. Since (2) is a special case of a lemma of Green [Proc. Roy. Soc. London Ser. A. **237** (1956), 574–581; MR **18**, 464] (lemma credited to Howarth, proof to Hall), Theorem (1) remains valid. *W. R. Scott.*

Baer, Reinhold. Classes of finite groups and their properties. Illinois J. Math. **1** (1957), 115–187.

This investigation is concerned with interrelationships of a wide variety of abstract group-theoretical properties of finite groups. Some properties that define a class θ of finite groups are inherited properties possessed by subgroups, homomorphic images, and direct products of groups in θ — for example, the property of being abelien, or of being soluble. Others are such that G is a θ-group if and only if $G/\Phi(G)$ is a θ-group, where the Frattini subgroup $\Phi(G)$ is the intersection of all maximal subgroups S of G. One important property of this sort is supersolubility, whereby G is called supersoluble if every minimal normal subgroup M of every homomorphic image $H=G/N$ of G is cyclic (and hence of prime order). For other classes of θ-groups G the structure of the corresponding minimal normal subgroups M of $H=G/N$ and of the automorphisms $m \to h^{-1}mh = m^h$ induced in them by H are stressed. A prominent role is played by the core S_G of a maximal subgroup S, namely its intersection S_G with all its conjugate subgroups in G. For example, if $S_G=1$, there exists at most one abelian normal subgroup $\neq 1$ of G, and at most two different minimal subgroups of G. If two exist they are isomorphic, and each the centralizer of the other. A section is devoted to the θ-commutator subgroup $[G, \theta]$ which is the intersection of subgroups X of G for which G/X belongs to θ.

The Σ-elements of a group G are those whose orders are divisible only by the primes in a given set Σ, and G is called Σ-closed if these elements form a subgroup (necessarily characteristic). This property obtains if and only if $G/\Phi(G)$ is Σ-closed, and is equivalent to other less easily stated properties involving minimal normal subgroups of factor groups. Furthermore a Σ-closed group G is called Σ-dissolved when its Σ-subgroup is soluble, and this property likewise requires the same of $G/\Phi(G)$. In several places the author has extended elementary structure properties studied by B. Huppert [Math. Z. **60** (1954), 409–434; MR **16**, 332]. The series of characteristic Fitting subgroups $F_n(G)$ is defined inductively by taking $F_0(G)=1$, and $F_{n+1}(G)/F_n(G)$ the product of normal nilpotent subgroups of $G/F_n(G)$. From any class of θ-groups another class θ' is defined whose θ-commutator groups are nilpotent; and, in particular, if $\Delta^0=1$ and $\Delta^{n+1}=(\Delta^n)'$, a sequence of classes is defined that is so related to the Fitting sequence that G is soluble only if

$G=F_n(G)$ for some n, and then G is a Δ^n-group, and so is $G/\Phi(G)$. Also each $H=G/N$ induces in each of its minimal normal subgroups M a Δ^{n-1}-group of automorphisms. For the results on dispersed groups, strictly dispersed groups, and many other details the reader is referred to the paper. Many properties of supersoluble groups are given in the final section. *J. S. Frame* (East Lansing, Mich.).

Wielandt, Helmut. Vertauschbare nachinvariante Untergruppen. Abh. Math. Sem. Univ. Hamburg **21** (1957), 55–62.

The subgroups in any chain $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_k=A$, in which each G_i is normal in its predecessor G_{i-1} are called "postnormal" (nachinvarianten) subgroups of G. The collection of abstract simple factor groups in a composition series connecting G to A is called $K(G, A)$, and arbitrary subsets of the set I of all simple groups are denoted by Λ, $\mathfrak{M}$, $\cdots$. The subgroup G_Λ is defined as the intersection of all postnormal subgroups C of G for which $K(G, C)\subseteq\Lambda$. Unions and intersections of postnormal subgroups of G are also postnormal in G. The first main theorem states that $(A\cup B)_\Lambda=A_\Lambda\cup B_\Lambda$. A collection of postnormal subgroups containing E forms a postnormal lattice N if it includes with A and B both $A\cap B$, $A\cup B$, and a normal chain connecting A with $A\cup B$. With N is associated the set Γ of all mappings α, β, $\cdots$ of N into itself. An important subset Ω of Γ consists of the so-called operators ω on N such that $(A\cup B)^\omega=A^\omega\cup B^\omega$, and if A is normal in B, so is A^ω normal in B. The mappings $A\to A_\Lambda$ and $A\to A_{\Lambda_1}\cap A_{\Lambda_2}\cap\cdots\cap A_{\Lambda_n}$ (where Λ_i are disjoint subsets of I) are examples of such operators. A general permutability theorem states that if all the factors ω_i in $\omega=\omega_1\omega_2\cdots\omega_n$ occur at least once in $\phi\psi$ then A^ω commutes with B if and only if A^ψ and B^ϕ commute. In particular, two postnormal subgroups A and B of G always commute if no prime exists which is the index of a normal subgroup of A and also of a normal subgroup of B. *J. S. Frame* (East Lansing, Mich.).

Mills, W. H. The automorphisms of the holomorph of a finite abelian group. Trans. Amer. Math. Soc. **85** (1957), 1–34.

An arbitrary finite abelian group G may be written $G=G'\times G^{(2)}$, where G' is of odd order, and $G^{(2)}$ is either trivial ($k=0$) or a direct product $G^{(2)}=C_1\times\cdots\times C_k$ of k cyclic groups $C_i=\{c_i\}$ of orders $n_1\geq n_2\geq\cdots\geq n_k>1$, each n_i a power of 2. The holomorph H of G is a semi-direct product of elements $(g, \sigma)=g\sigma^*$ built from its given normal subgroup G and a subgroup A^* of elements σ^* of H such that $\sigma^*g(\sigma^*)^{-1}$ is the image $\sigma\cdot g$ of g under the automorphism $\sigma\in A$ in the group of automorphisms A of G. This paper studies the group $\mathfrak{A}$ of all automorphisms of H, its subgroup $\mathfrak{B}$ that leaves G fixed, the subgroup $\mathfrak{C}$ that leaves each element of G fixed, the normal subgroup $\mathfrak{I}$ of inner automorphisms of H, the factor group $\theta=\mathfrak{A}/\mathfrak{I}$ of outer automorphisms of H, the center J of H (of order 1 or 2) consisting of the characteristic elements of G, and various related groups. First the author shows that $\mathfrak{B}=\mathfrak{C}\mathfrak{I}$. Next, for $\Gamma\in\mathfrak{C}$, the mapping $\Gamma\sigma^*=(\Gamma'\sigma)\sigma^*$ with $\Gamma'\sigma\in G$ defines a mapping of A into G such that $\Gamma'\sigma\tau=(\Gamma'\sigma)\sigma\cdot(\Gamma'\tau)$, so $\mathfrak{C}$ is found to be isomorphic with the group $Z^1(A, G)$ of one-dimensional cocycles, or crossed homomorphisms of A into G. Similarly $\mathfrak{I}\cap\mathfrak{C}$ is found to be the group $B^1(A, G)$ of coboundaries, and $\mathfrak{B}/\mathfrak{I}\cong\mathfrak{C}/\mathfrak{I}\cap\mathfrak{C}$ is the first cohomology group $H^1(A,G)$. Squares of elements in $\mathfrak{C}$ are shown to belong to $\mathfrak{I}\cap\mathfrak{C}$, so $H^1(A,G)$ is either trivial or a "DP2-group", which is the direct product of groups of order 2. Only if $n_1>n_2$ does the center J of H have an element $c_1^{n_1/2}$ other than identity. The group of homomorphisms $H(A, J)$ of A into J plays an important role in the detailed study of $H^1(A, G)$ and the outer automorphism group θ. At most four types of normal subgroups G^* of H exist that are isomorphic to G. The group θ is a DP2-group unless $n_1/2\geq n_2=k=2$, and is then the product of a DP2-group with the symmetric group of order 6 if $n_1=4$, or with the octic group if $n_1>4$. Bases are given for these groups in the various cases. *J. S. Frame* (East Lansing, Mich.).

Nagao, Hirosi. On the groups with the same table of characters as symmetric groups. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 1–8.

Knowing the character table of a finite group G implies a knowledge of the multiplication table of the classes of conjugate elements of G. In the case of S_n, it is possible to deduce from this multiplication of classes the orders of the elements in the classes defined by the partition (2^k) for $k\leq[\frac{1}{2}n]$ and, also the orders of all products of elements of the class with $k=1$. If we denote such elements of G by a_i ($i=1, 2, \cdots, n-1$), then the relations $a_i^2=1$, $(a_ia_j)^2=1$ with $j\neq 1+i$, $(a_ia_{i+1})^3=1$, which uniquely define S_n, are satisfied and G is isomorphic to S_n. *G. de B. Robinson* (Toronto, Ont.).

Steinberg, Robert. Prime power representations of finite linear groups. II. Canad. J. Math. **9** (1957), 347–351.

Dans la première partie [même J. **8** (1956), 580–591; MR **18**, 281], l'auteur avait montré l'existence de certaines représentations irréductibles pour les groupes classiques finis, le degré de ces représentations étant égal à l'ordre d'un p-groupe de Sylow du groupe considéré, où p est la caractéristique du corps de base servant à définir le groupe. Il obtenait ces représentations par des constructions géométriques dans lesquelles intervenait le fait qu'un p-group de Sylow, dans les groupes envisagés, apparaissait comme un groupe unipotent maximal. Or, dans son étude des groupes simples déduits des algèbres de Lie simples classiques (complexes), C. Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] a montré que c'est là un phénomène général pour tous ces groupes; il apparaissait donc vraisemblable que les résultats de l'auteur devaient pouvoir s'établir par une méthode uniforme pour tous les groupes finis considérés par Chevalley. C'est ce que fait l'auteur dans le présent travail, en s'appuyant de façon essentielle sur les résultats de Chevalley (et notamment sur la "décomposition de Bruhat" du groupe en doubles classes relatives à un groupe unipotent maximal U); il définit dans l'algèbre du groupe (sur un corps quelconque F) un idéal à droite dont la dimension est égale à l'ordre de U, et prouve que la représentation correspondante est irréductible si et seulement si la caractéristique de F ne divise pas l'indice du normalisateur de U. Il est à noter que pour le moment ces résultats ne recouvrent pas complètement ceux de la première partie, car certains des groupes considérés alors par l'auteur (notamment le groupe unitaire et le "second groupe orthogonal") ne rentrent pas parmi les groupes de la théorie de Chevalley. *J. Dieudonné* (Evanston, Ill.).

Curzio, Mario. Gli automorfismi del reticolo dei laterali dei sottogruppi d'un gruppo. Ricerche Mat. **4** (1955), 3–14.

Let S_G be the group of automorphisms of the lattice of cosets (for all possible subgroups) of a group G. It is

proved that if G is finite, then S_G is isomorphic to the symmetric group S_m, where m is the order of the group G, if and only if G is either cyclic of prime order or is the direct product of two groups of order two (the four-group). For an arbitrary element x of G let us denote by E_x the subgroup of all automorphisms from S_G which leave x fixed. If $x=1$, then in place of E_x we shall write E_G. It is proved that the subgroup E_x is normal in S_G if and only if $E_x=1$. There exists a one-one correspondence between the elements of the group G and the cosets of the group S_G with respect to the subgroup E_x. The group S_G contains a subgroup $\bar{G}$, isomorphic to G, such that $E_x \cap \bar{G}=1$ and $E_x \cup \bar{G}=S_G$ for arbitrary $x \in G$.

The following result is also proved: Let G be a finite special group; let $A_1, A_2, \cdots, A_t$ be its Sylow subgroups; let a_i be an arbitrary element of the group A_i; and let ω be an arbitrary element of the group E_G and α_i the corresponding element of the group E_{A_i}. Then $\omega(a_1 a_2 \cdots a_t)=\prod_{i=1}^{t} \alpha_i(a_i)$. *V. K. Turkin* (RŽMat **1957**, no. 1160).

Goffman, Casper. A lattice homomorphism of a lattice ordered group. Proc. Amer. Math. Soc. **8** (1957), 547–550.

Let L be the positive cone of an archimedean lattice ordered group G. It is proved that there exists a unique supremum-preserving lattice-homomorphism α of L, with kernel [0], satisfying $\alpha(x \cup y)=\alpha(x+y)$ for all $x, y \in L$, such that $\alpha(L)$ is the lattice of filets [cf., e.g., Jaffard, C. R. Acad. Sci. Paris **230** (1950), 1024–1025, 1125–1126; MR **11**, 579] in L. For a nonarchimedean G this is shown not always to hold. Also, the characterising properties of α are discussed. *A. Jaeger* (Cincinnati, Ohio).

Banaschewski, Bernhard. Über die Vervollständigung geordneter Gruppen. Math. Nachr. **16** (1957), 51–71.

There are classically two mathematical systems, ordered sets and uniform structures, which admit completion operations. For an ordered group, as is well-known, the order completion generally involves loss of some of the group structure, while the appropriate uniform structure for turning the group into a topological group is not always obvious. The author clarifies these matters in a masterly fashion.

Let G be an ordered (partially) group, and δG the completion of G by means of lower segments. Then δG is an ordered semigroup with maximal ordered subgroup G^*. A set $E \subset G$ is called a topological identity if (i) $\varepsilon \in E$ implies $\varepsilon>0$, (ii) for $\varepsilon, \varepsilon' \in E$ there is $\delta \leqq \varepsilon, \varepsilon'$ in E, (iii) for $\varepsilon \in E$ there is $\delta \in E$ with $2\delta \leqq \varepsilon$, (iv) for $\varepsilon \in E$, $x \in G$ there is $\delta \in E$ with $\delta \leqq x+\varepsilon-x$, (v) $\inf_G E=0$. Uniform structures are imposed only on groups which have topological identities; for each such E a subbase of neighborhoods of 0 is given by the sets $B_\varepsilon=[x: -\varepsilon \leqq x \leqq \varepsilon]$, $\varepsilon \in E$. This is called an order topology in G. A totally ordered group (dense in itself) has a unique order topology, the interval topology. In general, this is not true. The completion τG of G (for E) has the following properties: (a) τG is an ordered semigroup and $\tau G \subseteq G^*$, (b) if certain right and left uniformities defined in δG by E are equivalent, then τG is a group, (c) if G is totally ordered then $\tau G=G^*$, so that τG is a totally ordered group, (d) if G is a lattice ordered group then τG is a lattice ordered semigroup. More is true for abelian lattice-ordered groups in which $\inf_G [x: x>0]=0$. For example, G^* is complete for every order topology in G^*, and G^* is the complete hull of G (in a sense defined in the paper) with respect to all order topologies. The paper closes with examples which illuminate the phenomena involved. *C. Goffman* (Lafayette, Ind.).

Iizuka, Kenzo. Note on blocks of group characters. Kumamoto J. Sci. Ser. A. **2** (1956), 309–321.

This paper contains proofs of some of the results announced by R. Brauer in three notes [Proc. Nat. Acad. Sci. U.S.A. **30** (1944), 109–114; **32** (1946), 182–186, 215–219; MR **6**, 34; **8**, 14, 131]. Let G be a finite group, K a splitting field for G, Γ the group algebra of G over K and Γ^* the group algebra of G over a residue class field K^* of K determined by a prime ideal in K dividing a fixed rational prime p. The results fall into three categories. First the elementary divisors of the constituent C_τ of the Cartan matrix of G which is associated with a block B_τ of defect d are determined, and some results are obtained on the ordinary characters of G in K which belong to the block B_τ. Secondly the relationship between blocks of the subgroups of G and blocks of G are studied. In particular, the following theorem is proved. Let $\{H^\alpha\}$ be a complete set of representatives for the classes of conjugate subgroups of order p^d in G, and let N^α be the normalizer of H^α. Then all blocks of N^α have at least the defect d. Each block of defect d of N^α determines exactly one block of defect d of G, and each block of defect d of G is determined by a block of defect d of exactly one subgroup N^α. Finally the author proves some results on the central idempotents of Γ^* which are based on a paper by Osima [Math. J. Okayama Univ. **4** (1955), 175–188; MR **17**, 1182]. Osima's paper also contains proofs of several of the theorems of the present paper. R. Brauer has recently published a paper [Math. Z. **63** (1956), 406–444; MR **17**, 824] in which full details are given of much of the program announced in the notes listed above. *C. W. Curtis* (Madison, Wis.).

Mahler, K. A matrix representation of the primitive residue classes (mod $2n$). Proc. Amer. Math. Soc. **8** (1957), 525–531.

Let m and n be two integers satisfying the conditions $(m, n)=1$, $1 \leqq |m| \leqq n-1$, m is odd. Define $A(m, n)$ to be the n by n matrix (a_{hk}) where, if $m>0$, $a_{hk}=(-1)^q$ or 0 according as $0 \leqq i \leqq m-1$ or $i>m-1$, and i and q are determined by $km-h=nq+i$ with $0 \leqq i \leqq n-1$; while if $m<0$, $a_{hk}=(-1)^{r-1}$ or 0 according as $1 \leqq j \leqq -m$ or $-m+1 \leqq j \leqq n$, and j and r are determined by $-km+h=nr+j$ with $1 \leqq j \leqq n$. The author proves that the matrices $A(m, n)$ form an Abelian multiplicative group which is isomorphic to the group of primitive residue classes (mod $2n$) and that the isomorphism is defined by $A(m, n) \leftrightarrow \{m (\mathrm{mod}\ 2n)\}$. These matrices arose in connection with a discussion of plane symmetric convex domains of given lattice determinant and smallest area. *B. W. Jones* (Boulder, Colo.).

Jenner, W. E. On representations of the full linear group and some of its subgroups over infinite fields of prime characteristic. Monatsh. Math. **61** (1957), 157–160.

The author proves that, if k is an infinite field of characteristic $p>0$, then the full linear groups $\mathrm{GL}(n, k)$ and the special linear groups $\mathrm{SL}(n, k)$ for $n>1$, and the symplectic groups $\mathrm{Sp}(2n, k)$ for $n \geqq 1$ have rational irreducible representations of arbitrarily high degree. *S. A. Jennings.*

Landin, Joseph; and Reiner, Irving. Automorphisms of the general linear group over a principal ideal domain. Ann. of Math. (2) **65** (1957), 519–526.

Let R be a principal ideal domain with characteristic different from 2. Then R can be embedded in a division ring K. Let E be a right vector space over K with a basis

$e_1, \cdots, e_n$. Denote by $[E]$ the free right R-module $e_1R\oplus\cdots\oplus e_nR$. The group of invertible semi-linear maps of E onto itself is denoted by $\Gamma L_n(K)$, and the subgroup of this group consisting of the invertible semi-linear (relative to automorphisms of R) maps of $[E]$ onto itself is denoted by $\Gamma L_n(R)$. The subgroups of these groups consisting of linear maps are denoted by $\mathrm{GL}_n(K)$ and $\mathrm{GL}_n(R)$, respectively. Let E^* be the dual of E with dual basis $e_1^*, \cdots, e_n^*$ and $[\Sigma^*]$ the left R-module $Re_1^*\oplus\cdots\oplus Re_n^*$. For non-zero $\alpha\in K$, let λ_α be the homothetic map $x\to x\alpha$, $x\in E$, and let U be the group of those λ_α with α a unit in the center of R. Let χ be a homomorphism of $\mathrm{GL}_n(R)$ into U, with the property that $\chi(\lambda_\alpha)=\lambda_\alpha^{-1}$ if and only if $\alpha=1$. Then, for any $g\in\Gamma L_n(R)$, the map $u\to\chi(u)gug^{-1}$ defines an automorphism of $\mathrm{GL}_n(R)$. Similarly, if h is a one-to-one semi-linear map of $[E]$ onto $[E^*]$ relative to an antiautomorphism of R, then $u\to\chi(u)h^{-1}u^*h$, where u^* is the contragredient ${}^tu^{-1}$ of u, also defines an automorphism of $\mathrm{GL}_n(R)$. The principal result obtained in this paper is that, if $n\geqq 3$, then every automorphism of $\mathrm{GL}_n(R)$ is one of the above types. This extends to the case of a principal ideal domain the well-known results for division rings [Dieudonné, Mem. Amer. Math. Soc. no. 2 (1951); MR 13, 531]. The authors promise to deal later with $\mathrm{GL}_2(R)$. *C. E. Rickart.*

Szász, G.; und Szendrei, J. Über die Translationen der Halbverbände. Acta Sci. Math. Szeged **18** (1957), 44–47.

Let H be a semilattice (=idempotent commutative semigroup). A translation of H is function λ of H into itself satisfying $\lambda(xy)=\lambda(x)\cdot y$. It is shown that a function φ of H into itself is a translation if and only if $\varphi(x)\cdot y=\varphi(x)\varphi(y)$ (all x, $y\in H$). Also, the translations of H are shown to form a semilattice in which H is imbedded as an ideal. Further, in the ordering $x\leqq y\Leftrightarrow xy=y$, a closure operation λ is a translation if and only if $x<y\Rightarrow\lambda(x)\cdot y=\lambda(x)\lambda(y)$. These results contain some earlier results of G. Szász [same Acta **17** (1956), 165–169; MR **18**, 555]. *R. J. Koch* (Baton Rouge, La.).

Sade, A. Groupoïdes automorphes par le groupe cyclique. Canad. J. Math. **9** (1957), 321–335.

The author says, in effect, that a groupoid $(G, \times)$ is "automorphic by the cyclic group" provided $(G, \times)$ possesses a cyclic group A of automorphisms such that A is transitive and regular on the elements of G. In this case, G and A have the same cardinal. Actually, the author supposes that A is a specifically given cyclic group $(Q, +)$, either the additive group of integers or the additive group of integers mod n, and considers the problem of defining a single-valued binary operation $(\times)$ on Q such that the mapping $x\to x+t$ is an automorphism of $(Q, \times)$ for each t in Q. In view of the automorphism condition, multiplication $(\times)$ is completely defined by the function $f(x)=x\times 0$. Necessary and sufficient conditions that $(Q, \times)$ be a quasigroup are that the mappings $x\to f(x)$ and $x\to x-f(x)$ of Q be (I) one-to-one and (II) upon Q. {The author seems to be thinking entirely in terms of finite cyclic groups; in particular, he ignores (II), which is necessary in the infinite case.} Let us call f a "quasigroup" function if the above conditions are satisfied. The author considers operators a, b, c, D_h and E_h (where h ranges over Q) which replace a "quasigroup" function $f(x)$ by the "quasigroup" functions $x-f(x)$, $-f(-x)$, $f^{-1}(x)$, $f(x+h)$ and $f(x)+h$ respectively. These operators form a group; the subgroup generated by a, b, c is the dihedral group defined by $ab=ba$, $bc=cb$, $a^2=b^2=c^2=1$, $(ac)^3=(ca)^3=b$; moreover, $c^{-1}E_hc=D_h$ for each h. The operator D_h (or E_h) leads from a quasigroup defined by f to an isotopic quasigroup. Quasigroups of finite order n, automorphic by the cyclic group, exist for every odd n. (None exist for even n, but the author shows how to define analogous "incomplete" quasigroups of even order.) Complete details are stated without proof for odd n not exceeding 9. The rest of the paper is devoted to the theory of the subquasigroups and the automorphisms (other than the given translations $x\to x+t$) of a finite quasigroup automorphic by the cyclic group. {It seems to the reviewer that the author should have pointed out that a "quasigroup" function is otherwise known as a complete mapping of $(Q, +)$ and that his theory is akin to (but also distinct from) the theory of neofields. Pertinent references, among others, are two papers by Paige [Bull. Amer. Math. Soc. **53** (1947), 590–593; Duke Math. J. **16** (1949), 39–60; MR **9**, 6; **10**, 403].} *R. H. Bruck* (Madison, Wis.).

Almeida Costa, A. Über die Fastgruppentheorie. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 265–328.

The paper represents a reworking, with minor additions, of the theory of homomorphisms of a quasigroup upon a quasigroup and of the corresponding extension problems. {The author uses "Fastgruppe" instead of "Quasigruppe"; the latter term and its English derivative "quasigroup" seem to be standard usage.} Various authors have considered homomorphism theory. The earlier authors restricted the theory by introducing various inappropriate analogies to group theory. The first completely general theory (restricted, however, to finite quasigroups) was given by Garrison [Ann. of Math. (2) **41** (1940), 474–487; **47** (1946), 50–55; MR **2**, 7; **7**, 375]. Next Albert [Trans. Amer. Math. Soc. **54** (1943), 507–519; MR **5**, 229] gave the first general theory for arbitrary loops (i.e., quasigroups with an identity element.) And, finally, Kiokemeister [Amer. J. Math. **70** (1948), 99–106; MR **9**, 330] extended Garrison's work to arbitrary quasigroups. Garrison worked in terms of normal complexes (i.e., congruence classes); Albert made much use of the so-called multiplication groups (right-, left-, and two-sided) of a loop; and Kiokemeister worked directly in terms of congruence relations. The present author combines the viewpoints of Garrison and Albert (but refers, rather, to Albert and Kiokemeister.) He obtains the Kiokemeister theory with two minor additions. Let the subquasigroup P of the quasigroup Q be a normal divisor in the sense of Kiokemeister, so that the quotient quasigroup $Q'=Q/P$ exists uniquely and has P as an idempotent element. The author calls the normal divisor P a right normal divisor or a two-sided normal divisor of Q provided P is a universal right identity element or a two-sided identity element, respectively, of Q'. The restriction to right normal divisors (and, later, to two-sided normal divisors) allows some improvement in the various structural lemmas leading to a Jordan-Hölder theorem; for example, the intersection of a subquasigroup and a right normal divisor is never empty. The concluding part of the paper deals with the following extension problem: given quasigroups P, Q', where Q' has a universal right identity element, to construct all quasigroups Q (one Q always exists) such that P is a right normal divisor of Q and Q/P is isomorphic to Q'. Here the author adds very little to the rather embryonic theory of Albert [Trans. Amer. Math. Soc. **55** (1944), 401–419; MR **6**, 42] and the reviewer [ibid. **56** (1944), 141–199; MR **6**, 116]. Other topics considered are isotopy of quasigroups, quasi-

groups with the inverse property (here called "Bruck quasigroups"), and the centre of a quasigroup.

R. H. Bruck (Madison, Wis.).

See also: Partial Order, Lattices: Aubert. **Homological Algebra:** Higman; Tannaka. **Computing Machines:** Lloyd.

Homological Algebra

Auslander, Maurice. On regular group rings. Proc. Amer. Math. Soc. **8** (1957), 658–664.

A ring R (with identity) is regular if for each $x \in R$ there is a $y \in R$ such that $xyx=x$. Let G be a group, K a commutative ring with identity, and $K(G)$ the group ring of G with respect to K. The main theorems of this paper then state that, if G is locally finite, $K(G)$ is regular if and only if K is regular and is uniquely divisible by the order of each element in G. Moreover, if $K(G)$ is regular (G not necessarily locally finite), then G is a torsion group and K is a regular ring which is uniquely divisible by the order of each element of G.

The crux of the argument is in the homological characterization of regular rings. A ring R is regular if and only if $\mathrm{Tor}_1^R(R/J, R/I)=0$ for all right ideals J and left ideals I. By proving certain propositions about weak dimension of modules, and using the fact that Tor commutes with direct limits, the above main theorems are proved. The author also shows that if G is a commutative group and K a field whose characteristic does not divide the order of any torsion element in G, then the weak global dimension of $K(G)$ is equal to the rank of G.

A few minor typographical errors might be noted: p. 658, second line from bottom, read w. gl. dim $R<n\cdots$; Theorem 1(b), read $J\cap I=JI$; p. 659, line 12, read $\mathrm{Tor}_1^R(R/J, R/I)$. *D. Buchsbaum* (Providence, R.I.).

Higman, D. G. Induced and produced modules. Canad. J. Math. **7** (1955), 490–508.

Es sei A ein Ring mit 1, S ein Teilring von A, welcher 1 enthält. Einem S-Modul M (von rechts; 1 operiert identisch) wird ein "induziertes Paar" $(I(M), \kappa)$ zugeordnet: es besteht aus dem Tensorprodukt $M\otimes_S A$, als A-Modul aufgefasst, und dem S-Homomorphismus κ von M in $I(M)$, $\kappa(m)=m\otimes 1$. Dieses Paar ist maximal in dem Sinne, dass sich jeder S-Homomorphismus δ von M in einen A-Modul H über $I(M)$ faktorisieren lässt, $M\xrightarrow{\kappa} I(M)\xrightarrow{\beta} H$, wobei β ein durch δ eindeutig bestimmter A-Homomorphismus ist; durch diese Maximalität ist es bis auf Isomorphie charakterisiert. In dualer Weise ist das „produzierte Paar" $(P(M), \pi)$ erklärt: $P(M)$ ist der A-Modul $\mathrm{Hom}_S(A, M)$ und π der durch $\pi(f)=f(1)$, $f \in P(M)$, definierte S-Homomorphismus von $P(M)$ in M; es ist minimal (jeder S-Homomorphismus δ eines A-Moduls H in M lässt sich eindeutig über $P(M)$ faktorisieren, $H\xrightarrow{\beta} P(M)\xrightarrow{\pi} M$), und es ist durch diese Minimalität charakterisiert. — Ist M sogar ein A-Modul, so ist κ monomorph und π epimorph; $I(M)$ und $P(M)$ enthalten beide M als S-direkten (aber i.A. nicht A-direkten) Summanden.

Ist A der Gruppenring einer Gruppe G, S der Gruppenring einer Untergruppe U von endlichem Index in G, M ein S-Modul (d.h. ein Darstellungsmodul von U), so ist ebenfalls κ mono- und π epimorph, und ferner $I(M)\cong P(M)$; es ist der durch M induzierte Darstellungsmodul von G in der üblichen Terminologie der Darstellungstheorie. — Es wird allgemein die Frage untersucht, wann $I(M)\cong P(M)$ ist für alle S-Moduln M. Für einen Ring A, der über S eine (endliche) Links-Basis a_i und eine Rechts-Basis $\bar{a}_j$ besitzt, trifft dies genau dann zu, wenn A über S "selbst-dual" ist. Darunter wird verstanden, dass es in A eine invariante S-Bilinearform gibt; hier ist es damit gleichbedeutend, dass zu jeder Links-Basis a_j eine Rechts-Basis $\bar{a}_i$ existiert, derart dass aus $a_i a=\sum a_\beta s_{\beta i}$ folgt $a\bar{a}_i=\sum s_{ij}\bar{a}_j$, $s_{ij} \in S$. In diesem Falle wird ferner eine Bedingung dafür angegeben, dass ein A-Modul M in jedem A-Modul M', in welchem er S-direkter Summand ist, auch A-direkter Summand ist: notwendig und hinreichend dafür ist die Existenz eines S-Endomorphismus α von M mit $\sum \bar{a}_i\alpha a_i=$Identität. Diese Sätze verallgemeinern frühere Resultate ähnlicher Art des Verf. [Duke Math. J. **21** (1954), 369–376; MR **16**, 794], sowie von Eckmann [Proc. Nat. Acad. Sci. **39** (1953), 35–42; Ann. of Math. (2) **58** (1953), 481–493; MR **15**, 459, 397] und Gaschütz [Math. Z. **56** (1952), 376–387; MR **14**, 533].

Die allgemeinen Resultate werden angewendet (a) auf die Darstellungen von Frobenius-Algebren; (b) zur Herleitung eines Kriteriums für die Separabilität einer Algebra, in Anschluss an Hochschild [Ann. of Math. (2) **47** (1946), 568–579; MR **8**, 64]: zu jeder Links-Basis a_j existiert eine Rechts-Basis $\bar{a}_i$ wie oben, für welche ausserdem $\sum \bar{a}_i a_i=1$ ist; (c) zum Beweis der Sätze des Verf., von Eckmann und von Gaschütz (loc. cit.), über Darstellungen von Gruppen und Untergruppen von endlichem Index. *B. Eckmann* (Zürich).

Tannaka, Tadao. On the normal form of cohomology groups. J. Math. Soc. Japan **6** (1954), 16–31.

Let the finite abelian group G, with canonical decomposition $\{\sigma_1\}\times\cdots\times\{\sigma_r\}$, be a group of operators for the abelian group Ω. The present article considers certain n-dimensional cocycles ($n\geqq 1$) of the group G on Ω, called cocycles of standard form. Every standard cocycle f is completely determined by its "elementary values"

$$f(\sigma_{i_1}, N_{i_1}, \sigma_{i_1}, \cdots; \sigma_{i_2}, N_{i_2}, \cdots; \sigma_{i_k}, N_{i_k}, \cdots),$$

$$1\leqq i_1<i_2<\cdots<i_k\leqq r;\ k\geqq 1.$$

Here, for example, by $f(\sigma_1, N_1, \sigma_1; \sigma_2, N_2)$ is meant (written multiplicatively) $\prod_{ij} f(\sigma_1, \sigma_1^i, \sigma_1, \sigma_2, \sigma_2^j)$, $0\leqq i<\mathrm{ord}\,\sigma_1$; $0\leqq j<\mathrm{ord}\,\sigma_2$. It is shown that in every class of cohomologous cocycles there is contained a cocycle of standard form. Necessary and sufficient conditions are given in order that a given system of elements of Ω may coincide with the elementary values of a cocycle of standard form.

The case $n=3$, $r=2$ applied to fields leads to the theorem: if K/k is an abelian extension with Galois group $G=\{\sigma_1\}\times\{\sigma_2\}$, then

$$H^3(G, K^*)\cong[N_{K_1/k}K_1^*\cap N_{K_2/k}K_2^*]/N_{K/k}K^*,$$

where K_1, K_2 are subfields corresponding to the subgroups $\{\sigma_1\}$, $\{\sigma_2\}$ and K^*, K_1^*, K_2^* are the multiplicative groups of the corresponding fields.

Z. I. Borevič (RŽMat **1955**, no. 4902).

See also: Lie Groups and Algebras: Séminaire "Sophus Lie". **Algebraic Topology:** Hilton.

THEORY OF NUMBERS

General Theory of Numbers

Carlitz, Leonard. The number of points on certain cubic surfaces over a finite field. Boll. Un. Mat. Ital. (3) **12** (1957), 19–21.

In the present paper the author employs a straightforward method to derive explicit formulas for the number of solutions in a finite field GF(q) of the equation

(*) $ax^2+by^2+cz^2+2a'x+2b'y+2c'z=2dxyz+e.$

The general result is somewhat too involved to be given here. It can be mentioned, however, that in the particular case when $a'=b'=c'=0$ and $abcd\neq 0$ the number N of solutions x, y, z of (*) is given by

$$N=q^2+1+q(\psi(a)+\psi(b)+\psi(c)+\psi(e))\psi(d^2e-abc),$$

where q is odd and $\psi(a)=+1$, -1, 0 according as a is a square, a non-square, or zero of GF(q). The case $a'=b'=c'=0$, $ab=0$, $cd\neq 0$ is covered by a similar formula.

A. L. Whiteman (Los Angeles, Calif.).

Duma, N. On a problem of Hua Lo-Ken. Latvijas PSR Zinātņu Akad. Vēstis 1953, no. **1** (66), 159–162. (Russian. Latvian summary)

Some obvious conclusions are drawn from the theorem of Dirichlet on primes in an arithmetic progression.

Yu. V. Linnik (RŽMat **1953**, no. 30).

Brčić-Kostić, Mato. Solution of the diophantine equation $x^4+y^4=z^6$. Bull Soc. Math. Phys. Serbie **8** (1956), 125–136. (Serbo-Croatian. Esperanto summary)

Palamà, Giuseppe. Sull'equazione indeterminata $x_1^2+\cdots+x_n^2+y^2=(n+1)x_1\cdots x_n$ e su altre analoghe. Riv. Mat. Univ. Parma **7** (1956), 89–123.

The author considers the general equation in the title as an outgrowth of the equation

$$x^2+y^2+1=3xy$$

studied recently by Schinzel and Sierpiński [Matematiche Catania **10** (1955), 30–36; MR **17**, 711] whose solutions are $(x, y)=(U_{2r-1}, U_{2r+1})$, where U_k is the kth term of the Fibonacci sequence $U_1=1$, $U_2=1$, $U_k=U_{k-1}+U_{k-2}$. The same general method of solving the quadratic equation for the unknown x_n, requiring that the discriminant be a perfect square and proceeding inductively, is used throughout. For example, the equation

$$x_1^2+x_2^2+x_3^2+x_4^2+y^2=4x_1x_2x_3x_4$$

has as solution

$$(A_k, A_{k+1}, A_{k+2}, A_{k+3}, [A_{k+3}-A_{k+2}][A_{k+1}-A_k]),$$

where A_k is the kth term of the sequence

$$1, 1, 3, 11, 41, 153, 571, 2131, 7953, \cdots$$

in which $A_k=4A_{k-1}-A_{k-2}$. Numerous similar equations such as $x^2+y^2+20=7xy$ are solved in the same manner.

D. H. Lehmer (Berkeley, Calif.).

Sachs, Horst. Untersuchungen über das Problem der eigentlichen Teiler. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 223–259.

Let ξ be an algebraic number. The author first studies in detail the question for what values of $n=1, 2, \cdots$ there is a prime ideal $\mathfrak{p}$ in the algebraic number field $K(\xi)$ which is an intrinsic divisor of ξ^n-1, i.e., which satisfies $\mathfrak{p}|\xi^n-1$, $\mathfrak{p}\nmid\xi^m-1$ for $m=1, 2, \cdots, n-1$. This holds if and only if $\mathfrak{p}\nmid n$ and $\mathfrak{p}|\mathfrak{d}_n(\xi)=\text{g.c.d.}\{u_n(\xi)/u_{n/p_\sigma}(\xi)\}$, where p_σ runs through the rational prime factors of n and $u_m(\xi)=(x^m-y^m)t^{-m}$, $\xi=x/y$, $(x, y)=t$; further, $n|N(\mathfrak{p})-1$. By studying the divisors of $(n, \mathfrak{d}_n(\xi))$ and the order of magnitude of $N(\mathfrak{d}_n(\xi))$, the author derives sufficient conditions on ξ in order that for almost all n there be an intrinsic divisor of ξ^n-1; the exceptional n belong to a finite set given explicitly. In particular, he deals with real and imaginary quadratic fields and gives detailed results for the exponent $n=4$.

Next, for rational ξ, he studies the intrinsic divisors of the sequence $z_n(\xi)-2$, where $z_n(\xi)=2\cos(n \arccos \xi/2)=\{\frac{1}{2}(\xi+\sqrt{(\xi^2-4)})\}^n+\{\frac{1}{2}(\xi-\sqrt{(\xi^2-4)})\}^n$. Here the foregoing results can be applied, with ξ replaced by

$$\eta=\tfrac{1}{2}(\xi+\sqrt{(\xi^2-4)}).$$

Thus a refinement of a theorem given by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 265–276, 277–280; MR **14**, 1064] and by Ward [Ann. of Math. (2) **62** (1955), 230–236; MR **17**, 127] is obtained.

Finally, the author discusses the functional and number-theoretic properties of the two sequences of polynomials x^n, $z_n(x)$ respectively. He generally characterizes, by means of such properties, two sets of equivalent sequences of polynomials. The paper concludes with a bibliography containing 72 titles and mentioning 46 further papers.

G. G. Lekkerkerker (Amsterdam).

Širšov, V. M. On an estimate of a definite integral. Belorussk. Politehn. Inst. Sb. Nauč. Rabot. **1953**, no. 4, 13–24. (Russian)

Let

$$W_0(N)=\int_{\mathfrak{M}} L_\alpha^r \exp(-2\pi i\alpha N)d\alpha,$$

where

$$L_\alpha=\sum_{x=1}^{Q}\sum_{y=1}^{[Px^{-\nu}]}\exp 2\pi i\alpha x^m y^n,$$

the m, n, r, N being natural numbers and where $\mathfrak{M}$ is the set of points belonging to the union of so-called complementary intervals defined by conditions given in the article.

The author endeavors to prove that $W_0(N)\ll p^{r-n-\nu+\mu}$, which would make it possible to complete the proof of the generalized Waring theorem of the type considered in the article under review.

But the reviewer has been unable to follow all the details of the proof since, for example, there are certain obscurities in the definition of the concept of complementary interval: there is no exact definition of the magnitudes x, x_1, x_2, $\cdots$, x_r (there is a similar obscurity in an earlier article of the author in no. 2 of the same Sbornik.

N. G. Čudakov (RŽMat **1954**, no.1039).

Širšov, V. M. An asymptotic solution for the number of solutions in positive integers $x_1, y_1, \cdots, x_r, y_r$ of the inequality $x_1^m y_1^n+\cdots+x_r^m y_r^n\leq N$. Belorussk. Politehn. Inst. Sb. Nauč. Rabot. **1953**, no. 5, 240–242. (Russian)

Let $K_r(N)$ be the number of solutions of the inequality $x_1^m y_1^n+\cdots+x_r^m y_r^n\leq N$ in natural numbers with $m>n$.

Using induction on the parameter r, the author proves that

$$K_r(N)=\Gamma^r\left(1+\frac{1}{n}\right)\Gamma^{-1}\left(1+\frac{r}{n}\right)\zeta^r\binom{m}{n}N^{r/n}+O(N^{r/n-1/n-1/m}).$$

This result is an almost obvious generalization of known facts. *N. G. Čudakov* (RŽMat **1954**, no. 3200).

See also: Analytic Theory of Numbers: Yüh; Erdös. **Theory of Algebraic Numbers:** Dieter.

Analytic Theory of Numbers

Yüh, Ming-I. A note on an arithmetical function. Sci. Record (N.S.) **1** (1957), no. 2, 9–12.

Let $F(n)$ be the number of prime divisors of n, distinct or not, and let $C_2, \cdots, C_7$ denote certain well-defined constants. The reviewer proved [Duke Math. J. **23** (1956), 41–44; MR **17**, 588] that

$$(1)\qquad \sum_{\substack{2^{F(n)}\\ n\le x}} 2^{F(n)}=C_6x\log^2 x+C_7x\log x+O(x)$$

(tne argument used in the proof of reviewer's Theorem 3 does not give any sharper result) and

$$(2)\qquad \sum_{\substack{2^{F(n)}\\ n\le x}} 2^{F(n)}=C_2x\log x-C_5x+O(x^c)$$

for $c>c_0(=.84\cdots)$, and conjectured that (2) holds for any $c>d=(\log 2)/(\log 3)$. The author proves this conjecture. In his inference from (2) with $c=d+\varepsilon$ to a sharpened version of (1) with error term $O(x^{d+\varepsilon})$, the author repeats an oversight of the reviewer. In fact, the left-hand member of (1) has a discontinuity of saltus x whenever $x=2^k$, and the error term cannot be of lower order. The author asserts (without proof) also the validity of

$$(3)\ \sum_{n\le x} 2^{F(n)}=C_1x\log^2 x+C_2x\log x+C_3x+C_4x^d+O(x^{1/2+\varepsilon}).$$

Consideration of $x=2^k$ or $x=3^k$ shows that (3) cannot hold. Several misprints make the reading difficult.

E. Grosswald (Philadelphia, Pa.).

Yüh, Ming-I. On the differences between squarefree numbers. Sci. Record (N.S.) **1** (1957), no. 3, 13–16.

Combining van der Corput's method with an estimate of Vinogradov's, the author improves a former estimate by the reviewer [J. London Math. Soc. **29** (1954), 16–20; MR **15**, 289] for the difference $s_{n+1}-s_n$ of two consecutive squarefree numbers. However, Rankin in his paper on van der Corput's method of exponent pairs [Quart. J. Math. Oxford Ser. (2) **6** (1955), 147–153; MR **17**, 240] has pointed out that this method leads to an estimate, namely $s_{n+1}-s_n=O(n^{\gamma+\varepsilon})$, $\gamma=0.22198215\cdots$, which is superior to that given by the author. The paper is somewhat condensed, and there are some misprints.

H. E. Richert (Göttingen).

Fine, N. J. On a system of modular functions connected with the Ramanujan identities. Tôhoku Math. J. (2) **8** (1956), 149–164.

The author discusses modular functions connected with the Dedekind function $\eta(\tau)=x^{1/24}\prod_{m=1}^{\infty}(1-x^m)$, where $x=\exp 2\pi i\tau$. He starts from the identity

$$\psi(\tau)\equiv(-1)^{\lambda}\frac{\eta(\tau/q)}{\eta(q\tau)}=1+\sum_{k=1}^{\nu}W_k(\tau),$$

where $q=6\lambda\pm1$, $\nu=(q-1)/2$ and $W_k(\tau)$ is a power series in x multiplied $x^{6k^2/q-k}$ [cf. Atkin and Swinnerton-Dyer, Proc. London Math. Soc. (3) **4** (1954), 84–106; MR **15**, 685]. He then develops the properties of W_k under transformations of the subgroups $\Gamma_0(q)$ (defined by $q|c$), $\Gamma_0^0(q)$ ($q|b$, c) and $\Gamma(q)$ ($q|b$, c, $a-1$, $d-1$), where $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a modular substitution. This is done by the use of simple operations on infinite products and the transformation formula for one of the theta-functions.

Next, the functions $\psi_r(\tau)=\psi(\tau+r)$, $0\le r<q$, are studied and it is proved that

$$\psi_r(M\tau)=\psi_{rd^2+bd}(\tau),\quad M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(q).$$

This formula permits the derivation of the Ramanujan-Watson modular equation for $\eta(\tau/q)/\eta(\tau)$ in a surprisingly simple way. The proof of the modular equation for $q=5, 7$ is very much shorter than Watson's [J. Reine Angew. Math. **179** (1938), 97–128]. The author also works out the modular equation for $q=11$, which may be of some use in investigating Ramanujan's partition congruences to the moduli 11^a. He uses it to develop an identity for the modulus 11, similar to one of the reviewer's.

In the last two sections the author presents, as applications of the preceding theory, proofs of two of Ramanujan's unpublished and unproved identities, of which one is

$$49x\prod_{m=1}^{\infty}(1-x^{7m})^3(1-x^m)^3+8\prod_{m=1}^{\infty}\frac{(1-x^m)^7}{(1-x^{7m})}=$$
$$8-7\sum_{n=1}^{\infty}x^n\sum_{d\mid n}\left(\frac{d}{7}\right)d^3.$$

He also proves identities of Slater, Newman and Rademacher. *J. Lehner* (E. Lansing, Mich.).

Petersson, Hans. Über Partitionenprobleme in Verbindung mit Potenzresten nach einem Primzahlmodul. Math. Z. **66** (1956), 241–268.

Using the results of previous papers [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1954**, no. 2; Acta Math. **95** (1956), 57–110; MR **17**, 129, 1057], the author discusses a new partition function, $\pi_n(k, q, l)$, defined as follows. Let l, q be integers, $l\ge2$, q prime, $q\equiv1 \pmod{2l}$, $q>2l+1$. Let $\mathfrak{R}$ be the multiplicative group of residues modulo q which are prime to q, $\mathfrak{P}$ the subgroup of lth powers of elements in $\mathfrak{R}$, and $\mathfrak{B}$ a system of representatives of the cosets of $\mathfrak{P}$ in $\mathfrak{R}$. For each $s\in\mathfrak{B}$, let $k_s\ge0$, and $k=\sum k_s$ be positive. Then $\pi_n(k, q, l)$ is the number of partitions of the positive integer n into summands not divisible by q in which the summands from $s\mathfrak{P}$ ($s\in\mathfrak{B}$) appear in k_s "colors".

The generating function is essentially a modular function belonging to a subgroup $\Gamma^*[l, q]$ of the modular group which is defined by $b\equiv0 \pmod q$, $a\in\mathfrak{P}$ in $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma^*$. Thus the theorems of the above-mentioned papers enable the author to write the partition function π_n as a linear combination of convergent series (of the Rademacher type). He also gives asymptotic formulae; for example,

$$\pi_n=\frac{\omega(-k)}{2\sqrt{2\pi}}\rho_1^{\frac{1}{2}}n^{-3/4}\exp(\rho_1n^{1/2})$$
$$\times(1+c_1n^{-1/2}+c_2n^{-1}+c_3n^{-3/2}+O(n^{-1})),$$

where $\rho_1=2\pi\sqrt{[k(1-1/q)/6l]}$; c_1, c_2, c_3 are explicitly given rational functions of ρ_1 and μ; and μ and ω are explicitly given functions of $\{k_s\}$ and q.

In the second of the papers quoted above, Petersson proved an asymptotic formula for the quotient

$$\pi_n^+(q)/\pi_n^-(q)$$

of the number of partitions of n into quadratic residues and non-residues modulo q, respectively. In the present paper this is generalized. The author defines $\pi_n^{(\mathfrak{s})}(q)$ to be the number of partitions of n whose summands lie in the coset $s\mathfrak{P}$, $s \in \mathfrak{B}$; K is the subfield of $R(\exp 2\pi i/q)$ corresponding by the Galois theory to the subgroup $\mathfrak{P}$ of $\mathfrak{R}$. He proves:

$$\left| \log \frac{\pi_n^{(\mathfrak{s}t^{-1})}(q)}{\pi_n^{(t^{-1})}(q)} \right| = Rh(1+C_1(q)n^{-1/2}+O(n^{-1})),$$

where R and h are the regulator and class number, respectively, of K, the left member is a determinant in which s, t run over $\mathfrak{B}$, s, $t \neq 1$, and $C_1(q)$ depends on q, k, l and on certain L-series. *J. Lehner.*

Mitrović, Dragiša. Sur la fonction ζ de Riemann. C. R. Acad. Sci. Paris **244** (1957), 1602–1604.

A number of theorems on the kth derivatives of $\zeta(s)$ in the half-plane $s=\sigma+it$, $\sigma>1$, are stated without proof. For example,

$$|\zeta^{(k)}(s)| < \frac{k!}{(\sigma-1)^{k+1}}, \quad \left| \left(\frac{1}{\zeta(s)}\right)^{(k)} \right| < \frac{k!}{(\sigma-1)^{k+1}}$$

hold for $\sigma>1$, all t, and $k=1, 2, \cdots$. Finally, all the odd derivatives of $\zeta(s)-1/(s-1)$ are completely monotone in the interval $1<s<\infty$.

E. C. Titchmarsh (Oxford).

★**Prachar, Karl. Primzahlverteilung.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. x+415 pp. DM 55.00.

This is a comprehensive and up-to-date treatise covering most aspects of the theory of the distribution of primes. Its scope may be judged from the chapter headings. I. Elementary results; II. Sieve methods; III. The prime number theorem; IV. Primes in an arithmetical progression; V. Various applications; VI. The Goldbach problem; VII. Functiontheoretic properties of the L-functions. Explicit formulae and their applications; VIII. Trigonometric sums; IX. Theorems on the density of zeros of the L-functions and their application in prime number theory; X. The smallest prime in an arithmetical progression; Appendix; Literature; Index. Some remarks may be added by way of amplification. Chapter II is devoted to A. Selberg's modification of Brun's method, and deals only with upper bounds. Chapter III contains an analytical proof on classical lines, and a version of the elementary proof. Chapter VII includes a discussion of Ω-theorems, leading to Littlewood's theorem that $\pi(x) > \operatorname{li} x$ for arbitrarily large x. Various aspects of the problem of the difference between consecutive primes are discussed in V and IX. Chapter VIII includes an account of Vinogradov's method, with applications to the L-functions. Chapter X is devoted to the very difficult theorem of Linnik that the least prime in the arithmetical progression $kx+l$ $(0<l<k, (l, k)=1; x=0, 1, 2, \cdots)$ is less than k^C, where C is an absolute constant. General theorems not specifically concerned with primes are collected (usually with proofs) in an Appendix. This is a convenient arrangement, but it sometimes has the effect of making proofs in the body of the text appear deceptively short, since most of the work is done in the Appendix. The main text is supplemented by useful collections of examples at the ends of the chapters. These contain statements of further results, or indications of alternative proofs of results established in the relevant chapter. There are copious references to original sources.

The presentation usually follows the recent literature, but in some cases the author has preferred older methods. The treatment is full and clear, and in general accurate, but there is a curious lapse on pp. 145–146, where it is stated that Deuring [Math. Z. **37** (1933), 405–415] proved that the absence of zeros of $\zeta(\sigma+it)$ in the half-plane $\sigma>\frac{1}{2}$ implies the truth of Gauss's conjecture that there is only a finite number of negative fundamental discriminants $d=-k$ for which $h(d)=1$, where $h(d)$ is the number of classes of positive definite binary quadratic forms of discriminant d. It was in fact the contrary hypothesis (the existence of a zero in $\sigma>\frac{1}{2}$) that was proved by Deuring to have the stated consequence. (This was an implication in the opposite direction to an earlier one of Hecke, and it was by bringing the two together and bridging the gap between them that Heilbronn proved Gauss's conjecture.)

This book is a most valuable addition to the literature of prime number theory. Workers in this field owe a great debt of gratitude to the author for making such a wealth of material readily accessible in the relatively small compass of a little over 400 pages. The chapters need not be read consecutively, for, as the author points out in his preface, the material is so arranged that the reader can exercise considerable freedom in varying the order and selecting particular topics for study. *A. E. Ingham.*

Vinogradov, A. I. On an "almost binary" problem. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 713–750. (Russian)

This paper gives the detailed proof of a result already announced [Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 875–876; MR **17**, 349]. This is that if a sufficiently large number N is expressed in a fixed scale of notation as $N=a_0+a_1p+a_2p^2+\cdots+a_Rp^R$, then by changing an absolutely bounded number of the a_i by ± 1, one can derive a number N' which is a sum of two primes. (It is understood that a digit 0 can be changed only into 1 and a digit $p-1$ only into $p-2$.) A stronger result is proved on the assumption of the generalized Riemann hypothesis, namely, that the changes can be confined to $a_0, \cdots, a_M$, where $M=[(\log N/\log p)^{1/m}]$ and m is any fixed positive integer. The proofs, based on the method of Linnik [Mat. Sb. N.S. **32(74)** (1953), 5–60; MR **15**, 602] are long and intricate. *H. Davenport* (London).

★**Erdös, P. On additive arithmetical functions and applications of probability to number theory.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 13–19. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

An additive function $f(n)$ is one defined for positive integers n which takes real values and has the property that $f(mn)=f(m)+f(n)$ if m and n are relatively prime. The main problem discussed is that of the existence of a distribution function $\psi(c)$ such that the asymptotic density of the integers n for which $f(n)<c$ exists and equals $\psi(c)$. Many other problems arise from this. The present survey is partly complementary to that of M. Kac [Bull. Amer. Math. Soc. **55** (1949), 641–665; MR **11**, 161]. Besides giving an account of the considerable body of knowledge now attained on the distribution problem and

related problems, the author mentions many questions (both general and special) still awaiting an answer. Since 1954 further progress in the subject has been made by Kubilyus, whose article [Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 31–66; MR **18**, 17] should be read in conjunction with that of Erdös. Some other recent contributions are by H. Delange [see the paper reviewed below], and H. Halberstam [J. London Math. Soc. **31** (1956), 1–14, 14–27; MR **17**, 461]. *H. Davenport.*

Delange, Hubert. Sur les fonctions arithmétiques fortement additives. C. R. Acad. Sci. Paris **244** (1957), 1307–1309.

Let E be a set of positive integers of positive density $D(E)$, $f(n)$ a real strongly additive function defined on the positive integers, and define $A(f,x)=\sum_{p\leq x} f(p)/p$, $B(f,x)=\{\sum_{p\leq x} f^2(p)/p\}^{\frac{1}{2}}$ with p prime. The author states that the 'moment' method, as developed by the reviewer [J. London Math. Soc. **30** (1955), 43–53; MR **16**, 569] and by the author himself [Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 130–144; MR **17**, 946, 1437], may be extended to prove the following asymptotic formulae. If $B(f,x)\to\infty$ as $x\to\infty$ and $f(p)=o(B(f,p))$ as $p\to\infty$, then, for each fixed integer $q\geqq 0$,

$$(*)\qquad \lim_{x\to\infty} x^{-1}\sum_{n_0\leq n\leq x,\ n\in E}\left[\frac{f(n)-A(f,n)}{B(f,n)}\right]^q=D(E)\mu_q,$$

where μ_q is the qth moment of the Gaussian distribution, mean 0 and standard deviation 1, and E can be any one of a number of different sets. For instance, E may be an arithmetic progression, or it may be an arithmetic progression whose members obey the added condition that the number of prime factors of each member also lies in a certain arithmetic progression. For the latter case properties of Riemann's ζ-function and Dirichlet's L-functions in the critical strip are required. Let $N_E(f,x,t)$ be the number of positive integers $n\leq x$, $n\in E$, such that $f(n)\leqq A(f,x)+tB(f,x)$. The author points out that the mean value formulae (*) entail

$$\lim_{x\to\infty}\frac{1}{x}N_E(f,x,t)=D(E)(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{t}e^{-\frac{1}{2}u^2}du$$

[cf. Erdös and Kac, Amer. J. Math. **62** (1940), 738–742; MR **2**, 42]. *H. Halberstam* (London).

Delange, Hubert. Sur les fonctions arithmétiques fortement additives. C. R. Acad. Sci. Paris **244** (1957), 1604–1606.

The author states results which extend the theorems described in the preceding review to the simultaneous distribution of m linearly independent strongly additive functions $f_i(n)$ $(i=1,2,\cdots,m)$ with $n\in E$, where E is either the set of all positive integers, or one of the sets described in the note reviewed above. *H. Halberstam.*

Delange, Hubert. Sur les fonctions arithmétiques fortement additives. C. R. Acad. Sci. Paris **244** (1957), 2122–2124.

With the notation of the two preceding reviews, a recent theorem of Kubilyus [Vestnik Leningrad. Univ. **10** (1955), no. 11, 59–60; MR **17**, 588, but ignore last sentence of this review] states that if (i) $\lim_{x\to\infty} B(f,x)=+\infty$, (ii) for $\lambda>1$, $B(f,x^\lambda)\sim B(f,x)$ as $x\to\infty$, then setting

$$K_x(t)=B^{-2}(f,x)\sum_{f(p)\leq tB(f,x)}{}^{p\leq x} f^2(p)/p,$$

there exists a distribution function F such that

$$\lim_{x\to\infty} x^{-1}N(f,x,t)=F(t)$$

at all continuity points of F, if, and only if, there exists a function $K(t)$, non-decreasing in $(-\infty,\infty)$, with the property that $\lim_{x\to\infty}K_x(t)=K(t)$ at all continuity points of K. When this is the case, then for all real t,

$$\int_{-\infty}^{\infty}e^{itu}dF(u)=\exp\left\{\int_{-\infty}^{\infty}\frac{e^{itu}-1-itu}{u^2}\,dK(u)\right\}.$$

The theorem of Kubilyus generalises a well-known theorem of Erdös and Kac [Amer. J. Math. **62** (1940), 738–742; MR **2**, 42], and, indeed, contains as special cases all previous generalisations of the Erdös-Kac theorem. [See H. N. Shapiro, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 426–430; MR **18**, 113; Halberstam, J. London Math. Soc. **30** (1955), 43–53; MR **16**, 569; Delange, Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 130–144; MR **17**, 946, 1437.]

The author explains, without giving details, that one may replace the Vigo Brun method in the proof of Kubilyus' theorem by the moment method applied to a related strongly additive function. He further illustrates the effectiveness of the moment method in this subject by stating, without proof, that the mean value formulae of equation (*) [see the review second above] hold, subject to the conditions of Kubilyus' theorem and the one extra condition $f(p)=O(B(f,p))$ (and do not hold without some such extra condition), with μ_q equal to $\int_{-\infty}^{\infty}t^q dF(t)$. Thus the author's result stands in the same relation to Kubilyus' theorem as the reviewer's does to the Erdös-Kac theorem. *H. Halberstam* (London).

Theory of Algebraic Numbers

Albert, A. A. On certain trinomial equations in finite fields. Ann. of Math. (2) **66** (1957), 170–178.

This paper deals mainly with the solution of the trinomial equation $y^s+y^t=1$ in the field $\mathfrak{F}(x)$, where $\mathfrak{F}$ is a finite field of q elements, x an element of prime period $p>3$ such that q is primitive modulo p, and $y=x-1$. All solutions with s and t prime to p, called exceptional solutions, are determined. Through normalization of the equation $y^s+y^t=1$, the author finds that the solutions are completely determined by the solutions with both s and t even or odd, given as follows. When q is even:

(1) $s=q^{\mathrm{ind}\beta}-q^{\mathrm{ind}(\beta-\alpha)}$; $t=q^{\mathrm{ind}\alpha}-q^{\mathrm{ind}(\alpha-\beta)}$,

which coincide modulo the period of y with those given by

(2) $s=eq^{\mathrm{ind}\beta}-q^{\mathrm{ind}(\beta-\alpha)}+p\rho$; $t=eq^{\mathrm{ind}\alpha}-q^{\mathrm{ind}(\alpha-\beta)}+p\rho$,

where $e=q^{\mathrm{ind}2}-1$, $\rho=q^{\frac{1}{2}(p-1)}-1$, ind α denotes the index of α modulo p relative to q, and (α,β) ranges over the $\frac{1}{2}(p-1)(p-2)$ distinct combinations two at a time of the integers $1,2,\cdots,p-1$. When q is odd, (1) yields all exceptional solutions with s and t even, and (2) yields all exceptional solutions with s and t odd. Finally, these exceptional solutions are separated into classes relative to conjugacy defined by (s,t) conjugate to (s_1,t_1) if and only if $s_1=q^k s$, $t_1=q^k t$. In addition, the author also proves earlier in the paper that the trinomial equation $x^s+x^t=1$ is impossible in $\mathfrak{F}(x)$. *Shen Lin* (Columbus, O.).

Chamfy, Christiane. Valeur minima du module pour un ensemble fermé d'indices algébriques. C. R. Acad. Sci. Paris **244** (1957), 1992–1994.

In the notation of the paper by John B. Kelly [Amer. J. Math. **72** (1950), 565–572; MR **12**, 81], let S_1 denote

the set of algebraic integers θ whose conjugates (except θ itself) all have moduli less than one. Let S_2'' denote the set of all imaginary algebraic integers θ whose conjugates (except θ itself and one other $\theta' \neq \theta$) all have moduli less than 1. Kelly proved, among other things that S_1+S_2'' is closed. In the present paper, the author sketches proofs of the following: (1) If θ_1 is the zero greater than 1 of z^3-z-1, then $\sqrt{\theta_1}$ is the minimum of the moduli of numbers in S_2'', and this minimum is attained only by the zeros $\pm i\sqrt{\theta_1}$ of z^6-z^2+1 and by the conjugates of $\pm 1/\theta_1$, zeros of $z^3 \pm z^2 \mp 1$. (2) There does not exist a pair of conjugate numbers in S_2' (the set of all real algebraic integers whose conjugates, except for θ and one other $\theta' \neq \theta$, all have moduli less than 1) each with modulus less than or equal to $\sqrt{\theta_1}$ except for the pair $\pm\sqrt{\theta_1}$, zeros of z^6-z^3-1. *R. D. James* (Vancouver, B.C.).

Leopoldt, Heinrich-Wolfgang. Über ein Fundamentalproblem der Theorie der Einheiten algebraischer Zahlkörper. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 41–48 (1957).

The problem in question is to find an "effektiv-rational" procedure for determining a set of generators of the group of units of an algebraic number-field K over the rational field R, that is, a procedure entailing a finite number of rational operations applied to a basis for the integers of K. The author considers only the case in which K is normal over R with an abelian galois group, and he applies his earlier studies of such fields [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1953**, no. 2; MR **16**, 799] to reduce this case to that of determining bases for the "relative-units" of real cyclic fields Z over R. A relative-unit of Z over R is a unit whose norms relative to all proper subfields of Z are ± 1. The problem for the reduced case is solved by the author's two "finiteness" theorems whose proofs, merely sketched in the present paper, are to be given in full with examples elsewhere. The author obtains sharper results for the cases when the degree of the real cyclic field Z over R is $g=3$, 4 or 6. Equivalent results for the cases $g=3$ or 4 were used by Hasse [ibid. **1948**, no. 2; MR **11**, 503]. *R. Hull.*

Krasner, Marc. Prolongement analytique dans les corps valués complets: préservation de l'analycité par les opérations rationnelles. C. R. Acad. Sci. Paris **244** (1957), 1304–1306.

Krasner, Marc. Prolongement analytique dans les corps valués complets: préservation de l'analyticité par les opérations rationnelles; quasi-connexité et éléments analytiques réguliers. C. R. Acad. Sci. Paris **244** (1957), 1599–1602.

Krasner, Marc. Prolongement analytique dans les corps valués complets: uniformité des fonctions analytiques; l'analyticité des fonctions méromorphes. C. R. Acad. Sci. Paris **244** (1957), 1996–1999.

Krasner, Marc. Prolongement analytique dans les corps valués complets: préservation de l'analyticité par la convergence uniforme et par la dérivation; théorème de Mittag-Löffler généralisé pour les éléments analytiques. C. R. Acad. Sci. Paris **244** (1957), 2570–2573.

Krasner, Marc. Prolongement analytique dans les corps valués complets: démonstration du théorème de Mittag-Löffler; singularités au bord. C. R. Acad. Sci. **245** (1957), 270–274.

Continuing earlier notes [same C. R. **238** (1954), 2385–2387; **239** (1954), 468–470, 745–747; MR **16**, 799], and using the definitions in the review cited, the author defines two analytic elements to be equivalent if they can be connected by a chain of analytic elements in which adjacent elements coincide on a set containing a limit point. He defines an analytic function to be an equivalence class of analytic elements, and proves: Two equivalent analytic elements always take the same value at every point where both are defined (consequence of the ultrametric geometry). Analyticity is preserved by the rational operations. If the residue class field of k is not algebraically closed, analytic functions can be defined on k by embedding it in a K for which this is so, and the result is independent of K. Analytic functions have analytic derivatives, and a theorem on preservation of differential relations holds. The "Mittag-Löffler Theorem" states sufficient conditions (depending on the region of definition D) that an analytic element f be expressible as f_1+f_2, with f_1 defined by the principal parts of the poles of rational approximates to f and f_2 an analytic element with a larger region of definition; under further conditions, f_2 is a constant. *G. Whaples.*

Krasner, Marc. Prolongement analytique dans les corps valués complets: démonstration du théorème de Mittag-Löffler; singularités au bord. C. R. Acad. Sci. Paris **245** (1957), 1285–1288.

Remarque de l'auteur: Le présente texte remplace celui de C.R. Acad. Sci Paris **245** (1957), 270–274 [analysé ci-dessus], rendu incompréhensible par une erreur dans l'ordre des pages.

G. Whaples (Bloomington, Ind.).

Dieter, Ulrich. Beziehungen zwischen Dedekindschen Summen. Abh. Math. Sem. Univ. Hamburg **21** (1957), 109–125.

The Dedekind sums $s(d, c)$ investigated in this paper are defined by

$$s(d,c)=\sum_{k \bmod c}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{kd}{c}\right)\right),\quad c>0,$$

where $((x))=0$ or $x-[x]-\frac{1}{2}$ according as x is or is not an integer. The major theorem is as follows. Let

$$\tau_1=\frac{a_3\tau_3+b_3}{c_3\tau_3+d_3},\quad \tau_3=\frac{a_2\tau_2+b_2}{c_2\tau_2+d_2},\quad \tau_2=\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1},$$

be three modular transformations with coefficients c_1, c_2, $c_3>0$ satisfying the relation

$$\varepsilon\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix}\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\quad(\varepsilon=\pm 1).$$

Then

$$s(d_1,c_1)+s(d_2,c_2)+s(d_3,c_3)=\frac{\varepsilon}{4}-\frac{\varepsilon}{12}\left(\frac{c_1}{c_2c_3}+\frac{c_2}{c_3c_1}+\frac{c_3}{c_1c_2}\right).$$

Two important and interesting consequences may be cited here. The first is Rademacher's extension [Duke Math. J. **21** (1954), 391–397; MR **16**, 14] of the Dedekind reciprocity formula for $s(h, k)$. The second is the following theorem. Let p, d, t be integers such that $(d, t)=1$ if $p\equiv 0 \pmod t$ and $d^2 \mp pd \equiv 1 \pmod t$ if $p \not\equiv 0 \pmod t$. Then

$$s(pd \pm 1, pt)=\pm\frac{p^2+2-3pt}{12pt}.$$

The last formula implies a number of similar relations due to Rademacher [Math. Z. **63** (1956), 445–463; MR **18**, 114]. The author's techniques are mostly analytical in character, but in some instances alternative arithmetical proofs are given. *A. L. Whiteman.*

Nakano, Noboru. Über das Produkt von Primäridealen im unendlichen algebraischen Zahlkörper. J. Sci. Hiroshima Univ. Ser. A. **18**, (1954), 129–136.

In the principal order of an infinite algebraic number field, the author classifies all primary divisors $\mathfrak{q}$ belonging to a prime ideal $\mathfrak{p}$ into four classes: (i) $\mathfrak{qp}\neq\mathfrak{q}$ and $\mathfrak{q}\neq\mathfrak{q}:\mathfrak{p}$, (ii) $\mathfrak{qp}\neq\mathfrak{q}$ and $\mathfrak{q}=\mathfrak{q}:\mathfrak{p}$, (iii) $\mathfrak{qp}=\mathfrak{q}$ and $\mathfrak{q}\neq\mathfrak{q}:\mathfrak{p}$, and (iv) $\mathfrak{qp}=\mathfrak{q}$ and $\mathfrak{q}=\mathfrak{q}:\mathfrak{p}$. He proves that the product of primary ideals of type (ii) belonging to the same prime ideal is of type (ii), and similarly (iii)×(ii)=(iii), (iii)×(iii)=(iii), (iv)×(ii)=(iv), (iv)×(iii)=(iv), and (iv)×(iv)=(iii) or (iv). [See Nakano, same J. **17** (1954), 321–343; MR **16**, 907.] *Y. Kawada* (Tokyo).

★ **Nakano, Noboru. Idealtheorie in unendlichen algebraischen Zahlkörpern.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 249–251. Science Council of Japan, Tokyo, 1956.

The author studies ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots$ in the ring $\mathfrak{O}$ of all integral algebraic numbers of a field K which is the union of a denumerable infinity of fields $K_0 \subset K_1 \subset \cdots \subset K_\nu \subset \cdots$, where K_0 is the rational field and each K_ν is of finite degree over K_0. He refers to several of his earlier papers on the ideal theory of $\mathfrak{O}$ [J. Sci. Hiroshima Univ. Ser. A. **15** (1952), 171–175; **17**, 11–20 (1953), 321–343 (1954); **18** (1955), 257–269, 271–287; MR **15**, 510; **16**, 907; **19**, 18; and the paper reviewed above] and states some conditions under which $\mathfrak{a}=\mathfrak{ba}$ or $\mathfrak{a}:\mathfrak{b}=\mathfrak{a}$, as well as conditions for the existence of $\mathfrak{c}$ such that $\mathfrak{a}=\mathfrak{bc}$. *R. Hull.*

Gundlach, Karl-Bernhard. Ganze Nichtspitzenformen der Dimension −1 zu den Hilbertschen Modulgruppen reell-quadratischer Zahlkörper. Arch. Math. **7** (1957), 453–456.

In a previous paper [Math. Z. **64** (1956), 339–352; MR **18**, 195] the author has treated Poincaré and Eisenstein series defined on a series of groups related to the Hilbert modular group. In the present paper the author proves the following theorem: Every entire modular form of dimension −1 on $\Gamma(1, \mathfrak{c})$ is the sum of a linear combination of Eisenstein series of dimension −1 and a cusp form. Here $\Gamma(1, \mathfrak{c})$ is the group of unimodular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, d integers in K, $b\equiv 0 \pmod{\mathfrak{c}}$, $c\equiv 0 \bmod \mathfrak{c}^{-1}$; K is a totally real algebraic field and $\mathfrak{c}$ is an ideal in K.
J. Lehner (E. Lansing, Mich.).

Lamprecht, Erich. Zur Eindeutigkeit von Funktional-primdivisoren. Arch. Math. **8** (1957), 30–38.

Suppose that A is a separably generated field of algebraic functions of one variable with K for its precise field of constants. Let M denote a set of discrete rank 1 valuations v_p of K such that each non-zero element of K is a unit for all but a finite number of v_p's. A discrete rank 1 prolongation v_P of v_p to A is called a functional prolongation if there exists an element $x \in A-K$ such that $v_P(\sum_{i=0}^{n} a_i x^i)=\min_i v_p(a_i)$, with $a_i \in K$. Further, denote by KP and AP the respective residue class fields of K and A for P. Suppose that $g(A/K)$ denotes the genus of A/K and $G(A/K)[\leq g(A/K)]$ the genus of $A\bar{K}/\bar{K}$, $\bar{K}$ the algebraic closure of K. The author terms a functional prime divisor P regular [quasi-regular] if (i) KP is the precise field of constants of AP, (ii) P is inert over some $K(y)$, $y \in A-K$, and (iii) $G(A/K)=G(AP/KP)$ [$g(A/K)=g(AP/KP)$]. Using methods resembling those employed for proofs of Gauss' Lemma and a module-theoretic argument of Deuring [Math. Z. **47** (1942), 643–654; MR **7**, 362] it is shown that for A/K with $G(A/K)\geqq 2$ there exists for all but a finite number of $v_p \in M$ a uniquely determined functional continuation to A which is regular [quasi-regular] such that y is a separating variable of A/K, and if there is a regular continuation for conservative A/K, then it is also quasi-regular. *O. F. G. Schilling.*

Terada, Fumiyuki. Complex multiplication and principal ideal theorem. Tôhoku Math. J. (2) **6** (1954), 21–25.

Let $\mathfrak{m}$ be an integral ideal of the imaginary quadratic field Ω, relatively prime to the discriminant of Ω and decomposable into a product of prime ideals of first degree. In the absolute field of classes K over Ω we have $\mathfrak{m}=(\gamma)$, $\gamma \in K$. In the case $N(\mathfrak{m})\equiv 1 \pmod{12}$ Hasse gave an explicit formula for the number γ, using complex multiplication of elliptic functions [Monats. Math. Phys. **38** (1931), 315–322]. In the present article, following the method of Hasse, the author sets up an explicit formula for γ in the case $N(\mathfrak{m})\equiv 5 \pmod{12}$.
Z. I. Borevič (RŽMat **1955**, no. 4851).

See also: General Theory of Numbers: Sachs.

Geometry of Numbers

Sawyer, D. B. The product of two linear forms whose values are restricted. J. London Math. Soc. **32** (1957), 213–217.

Let $L=\alpha u+\beta v+\rho$, $M=\gamma u+\delta v+\sigma$ be non-homogeneous linear forms in u, v with real coefficients such that $D=\alpha\delta-\beta\gamma\neq 0$. Then Minkowski's theorem states that $|LM|\leqq\frac{1}{4}|D|$ for some integers u, v. The author proves that integers u, v exist such that $|LM|\leqq(1/\sqrt{5})|D|$ and either $L>0$ or $M>0$, and further that integers u, v exist such that $|LM|\leqq\frac{1}{8}|D|$ and $L+M>0$. In his proof the author uses an idea of Rogers [same J. **29** (1954), 133–143; MR **15**, 941]; his (geometric) method also leads to a proof of Minkowski's theorem stated above. *J. F. Koksma.*

Steuerwald, Rudolf. Über die Perioden regelmässiger Kettenbrüche für Quadratwurzeln aus ganzen Zahlen. II. Math. Z. **67** (1957), 181–187.

The author continues the study of a problem begun in Part I [Math. Z. **52** (1950), 686–697; MR **12**, 393]. There are new inequalities and a classification of the solutions in the case $Q=3$. *H. S. Wall* (Austin, Tex.).

★ **Cassels, J. W. S. An introduction to Diophantine approximation.** Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Cambridge University Press, New York, 1957. x+166 pp. $4.00.

Chapter headings: I. Homogeneous approximation; II. The Markoff chain; III. Inhomogeneous approximation; IV. Uniform distribution; V. Transference theorems; VI. Rational approximation to algebraic numbers; VII. Metrical theory; VIII. The Pisot-Vijayaraghavan numbers; and Appendices.

This tract is rather more than an introduction to the subject of Diophantine approximation, and is best described as a concise textbook. It contains a remarkable

wealth of material, and most of the major theories of the subject are treated fairly fully. The author has taken much trouble to present the proofs in small compass, though in consequence it is not always possible to follow the general principle of a proof without mastering the details.

Chapter II contains all the main results of Markoff's theory of the minima of indefinite binary quadratic forms, proved in a mere 25 pages. Chapter III gives a full account of what is known in connection with Kronecker's theorem. Chapter IV contains the substance of Weyl's great memoir of 1916, in a modern presentation. Chapter V is based on the work of Khintchine, with recent developments by Hlawka, the author and Birch. Chapter VI contains the proof of K. F. Roth's remarkable recent theorem. Chapters VII and VIII are well-integrated accounts of work which was previously scattered in the literature.

Everyone interested in the subject has reason to be grateful to the author for this valuable work. It should be studied in conjunction with Koksma's report [Diophantische Approximationen, Springer, Berlin, 1936], which surveys a somewhat wider field, but for the most part without proofs. *H. Davenport* (London).

Cassels, J. W. S. On the sums of powers of complex numbers. Acta Math. Acad. Sci. Hungar. **7** (1956), 283–289. (Russian summary)

Let $b_1, \cdots, b_k$ be positive numbers, let $z_1, \cdots, z_k$ be complex, and put $s_\nu = b_1 z_1^\nu + \cdots + b_k z_k^\nu$. In the case $b_1 = \cdots = b_k = 1$, it is shown that if $\max_{1 \le j \le k} |z_j| \ge 1$, then

$$(*) \qquad \max_{1 \le \nu \le 2k-1} |s_\nu| \ge 1.$$

Equality holds in (*) if $z_1 = 1$, $z_2 = \cdots = z_k = 0$, and it is shown that the range of the max in (*) cannot be decreased to $1 \le \nu \le 2k-2$. [This was known earlier; see P. Turán, Eine neue Methode in der Analysis und deren Anwendungen, Akad. Kiadó, Budapest, 1953; MR **15**, 688.] In the general case, it is shown that, always $\operatorname{Re} s_\nu \ge 0$ for some ν with $1 \le \nu \le 4^k - 1$, but that, for suitable b_j and z_j, $\operatorname{Re} s_\nu < 0$ for all ν with $1 \le \nu \le 3^k - 1$. Finally, an analytic proof is given of a weakened version of Dirichlet's theorem: if $q > q_0(n)$ and if $\vartheta_1, \cdots, \vartheta_n$ are any real numbers, then there is an integer ν with $1 \le \nu \le q^n$ such that, for suitable integers e_j, $|\vartheta_j \nu - e_j| \le c(\log q)^{1/2}/q$ $(j = 1, \cdots, n)$, where c depends only on n. The fact that Fejér's kernel

$$\tfrac{1}{2} + \sum_{\nu=1}^{N} \left(1 - \frac{\nu}{N+1}\right) \operatorname{Re} z^\nu$$

is non-negative for all positive integers N and all complex z with $|z| \ge 1$ plays an important role in the proofs.
W. J. LeVeque (Ann Arbor, Mich.).

Turán, P. Remark on the preceding paper of J. W. S. Cassels. (Application to approximative solution of algebraic equations.) Acta Math. Acad. Sci. Hungar. **7** (1956), 291–294. (Russian summary)

Inequality (*) of the preceding review paper is used to give the following estimate for the error in the approximation of the largest root of a polynomial equation by the Bernoulli-Lobačevski-Graeffe method. Let $f_0(x) \equiv x^n + a_{10}x^{n-1} + \cdots + a_{n0} = 0$ be the equation to be solved, with roots $z_1, \cdots, z_n$, where $|z_1| \ge \cdots \ge |z_n|$. For $\nu \ge 1$ put $f_k(x) = (-1)^n f_{k-1}(x^{1/2}) f_{k-1}(-x^{1/2})$, and let s_{jk} be the sum of the jth powers of the zeros of $f_k(x)$, computable from the Newton-Girard formulas. Then

$$(1/n)^{2^{-k}} \le |z_1| (\max_{1 \le j \le 2n-1} |s_{jk}|^{1/j})^{-2^{-k}} \le 1.$$

W. J. LeVeque (Ann Arbor, Mich.).

Obrechkoff, N. Sur une question de l'approximation diophantique des formes linéaires. C. R. Acad. Bulgare Sci. **9** (1956), no. 4, 1–4. (Russian summary)

In this article it is established that the constant of Borel [J. Math. Pures Appl., **9** (1903), 329–375] in questions of diophantine approximation of linear forms is exactly equal to unity. (Translation of the Russian summary.)
H. A. Antosiewicz (Providence, R.I.).

Reeve, John E. Le volume des polyèdres entiers. C. R. Acad. Sci. Paris **244** (1957), 1990–1992.

Let l denote the lattice of points in the plane E_2 with integral coordinates. An l-polygon is defined as a non-empty subset of E_2 which admits a finite simplicial covering by rectilinear simplices having vertices in l and such that each simplex of dimension 0 or 1 belongs to at least one 2-simplex. The author draws attention to a formula for the area $A(\gamma)$ of an l-polygon γ, with boundary $\bar{\gamma}$:

$$(*) \qquad A(\gamma) = \{l(\gamma) + N(\gamma)\} - \tfrac{1}{2}\{l(\bar{\gamma}) + N(\bar{\gamma})\},$$

where $l(\gamma)$, $l(\bar{\gamma})$ denote the number of points of l belonging to γ and $\bar{\gamma}$, and where $N(\gamma)$, $N(\bar{\gamma})$ are the Euler-Poincaré characteristics of γ and $\bar{\gamma}$, respectively. (For an arbitrary finite simplicial complex composed of α_i i-simplices, $0 \le i \le n$, the Euler-Poincaré characteristic is defined here as $\sum_{i=1}^{n} (-1)^{i+1} \alpha_i$.) This is a generalization of the well-known formula $A(\gamma) = l(\gamma) - \frac{1}{2} l(\bar{\gamma}) - 1$, valid when $\bar{\gamma}$ is a closed Jordan curve. Other generalizations have been published by E. Ehrhart and numerous references are given.

A 3-dimensional analogue of (*) is stated; the proof of this and other results are due to appear later in the Proc. London Math. Soc. *J. H. H. Chalk* (Hamilton, Ont.).

See also: Groups and Generalizations: Mahler. **Functions of Complex Variables:** Denjoy.

ANALYSIS

Functions of Real Variables

Sternberg, Shlomo; and Wintner, Aurel. On a class of analogies between differential equations and implicit equations. J. Analyse Math. **5** (1956/57), 34–46.

The existence and properties of solutions near (0, 0) of the equation (1) $y = xf(x, y)$, $y(0) = 0$ are studied with particular reference to analogies with the differential equation (2) $dy/dx = f(x, y)$, $y(0) = 0$. f is supposed to be real-valued and continuous on $0 \le x \le a$, $-b \le y \le b$ and bounded by M, and solutions are considered for $0 \le x \le c = \min(a, b/M)$. For each x in $(0, c)$ a solution of (1) exists; any solution has derivative $f(0, 0)$ at 0. Apart from this, functions f can be found such that the possible solutions $y(x)$ fill an arbitrary set. A sufficient condition that the solution be unique in some interval near 0 is that $f(x, y') - f(x, y'') < |y' - y''|/|x|$ for x in a small positive interval and y', y'' in a small interval about 0; another is that f be strictly decreasing in y; unlike (2), (1) has not a unique solution for a decreasing f. If f increases, the

maximum possible solution of (1) increases, the minimum decreases. If there is a unique solution of (1), then if y' is a solution of (1) with f' replacing f, $y'(x)\to y(x)$ as $f'\to f$ over $(0, c)$. Convergence of sequences defined by $y_{n+1}=xf(x, y_n)$ to solutions are also considered.

J. L. B. Cooper (Cardiff).

Dzyadyk, V. K. Continuation of functions satisfying a Lipschitz condition in the L_p metric. Mat. Sb. N.S. **40(82)** (1956), 239–242. (Russian)

A construction is given for extending a function ϕ of class $\mathrm{Lip}(\alpha, p)$, $0<\alpha\leq 1$, $p\geq 1$, on an interval to a function of the same class on the whole line. If domain $\phi=[0, a]$, an extension of the required type is the even function F whose value at x is $\phi(x)$ on $[0, a]$, $(a+1-x)\phi[a(a+1-x)]$ on $[a, a+1]$, and 0 on $[a+1, \infty)$. *A. E. Livingston.*

Nikol'skiĭ, S. M. On continuation of functions of several variables with preservation of differentiability properties. Mat. Sb. N.S. **40(82)** (1956), 243–268. (Russian)

If $r=(\rho_1, \cdots, \rho_n)$ is a vector of real Euclidean n-space R^n, let $|r|$ and $\|r\|$ denote $(\sum \rho_i^2)^{1/2}$ and $\sum |\rho_i|$, respectively; when ρ_i $(i=1, 2, \cdots, n)$ is a non-negative integer and f is a function whose value at $x=(\xi_1, \cdots, \xi_n)\in R^n$ is $f(x)$, write $f^{(r)}$ for $\partial^{\|r\|}f/\partial\xi_1^{\rho_1}\cdots\partial\xi_n^{\rho_n}$. For $0<\alpha<1$, $1\leq p\leq\infty$, Ω a region of R^n, $0<M<\infty$, ρ a non-negative integer, and $r=(\rho_1, \cdots, \rho_n)$ with ρ_i $(i=1, 2, \cdots, n)$ a positive non-integral real number, write (i) $f\in \mathrm{Lip}(\alpha, p; \Omega, M)$ whenever $\|f(\cdot+h)-f\|_{L_p(\Omega_1)}\leq M|h|^\alpha$ for all regions Ω_1 and all $h\in R^n$ such that $\bar{\Omega}_1\subset\Omega$ and $\Omega_1+[0, 1]h\subset\Omega$; (ii) $\phi\in W_p^{(\rho)}(\Omega, M)$ whenever $\phi^{(t)}\in L_p(\Omega)$ for $\|t\|\leq\rho$ and $\|\phi^{(s)}\|_{L_p(\Omega)}\leq M$ for $\|s\|=\rho$; (iii) $F\in W_p^{(\rho)}H^{(\alpha)}(\Omega, M)$ whenever $F^{(t)}\in \mathrm{Lip}(\alpha, p; \Omega, M)$ for $\|t\|=\rho$ and there exists a K such that $F\in W_p^{(\rho)}(\Omega, K)$; and (iv) $\Phi\in H^{(r)}(\Omega, M)$ whenever $\partial^k\Phi/\partial\xi_i^k\in L_p(\Omega)$ $(k=0, 1, \cdots, [\rho_i]; i=1, 2, \cdots, n)$ and $\partial^{[\rho_i]}\Phi/\partial\xi_i^{[\rho_i]}\in \mathrm{Lip}(1, p; \Omega, M)$ for $i=1, 2, \cdots, n$.

Let G be a bounded region of R^n with boundary a $(\rho+2)$-times differentiable surface S. If n_P is the (unique) interior normal to S at $P\in S$, let G_δ denote the set of points Q of R^n such that $Q\notin\bar{G}$ and $Q=P+\tau n_P$ for some $P\in S$ and some τ, $-\delta<\tau<0$; take δ sufficiently small so that this representation of $Q\in G_\delta$ is unique. If domain $f=\bar{G}$, then an extension of f to $\bar{G}\cup G_\delta$ is effected by the function f_δ for which $f_\delta(x)=f(x)$ when $x\in\bar{G}$ and $f_\delta(Q)=\sum_{\nu=0}^{\rho}\lambda_\nu f(P-\tau\gamma_\nu n_P)$ when $Q=P+\tau n_P\in G_\delta$, where $0<\gamma_0<\gamma_1<\cdots<\gamma_\rho\leq 1$ and $\sum_{\nu=0}^{\rho}(-\gamma_\nu)^\beta\lambda_\nu=1$ for $\beta=0, 1, \cdots, \rho$.

For $(p, \rho)=(\infty, 0)$ results of Kirszbraun [Fund. Math. **22** (1934), 77–108] and Czipszer and Gehér [Acta Math. Acad. Sci. Hungar. **6** (1955), 213–220; MR **17**, 136] show that if $f\in W_p^{(\rho)}H^{(\alpha)}(G, M)$, then there exist δ, M_1, and c such that $f_\delta\in W_p^{(\rho)}H^{(\alpha)}(G\cup G_\delta, M_1)$ and

$$M_1+\|f_\delta^{(t)}\|_{L_p(G\cup G_\delta)}\leq c(\|f\|_{L_p(G)}+M),$$

with c independent of $M+\|f\|_{L_p(G)}$. Dzyadyk [see the paper reviewed above] has proved this result for $1\leq p\leq\infty$ and $\rho=0, 1, 2, \cdots$ when $G\subset R^1$. The author here shows the statement to be valid for $G\subset R^n$, $1\leq p\leq\infty$, and $\rho=0, 1, 2, \cdots$. For the case of a parallelepiped Δ of R^n, with sides parallel to the coordinate axes, he is able to prove that an $f\in H_p^{(r)}(\Delta, M)$ has an extension f_1 to a parallelepiped Δ_1, with sides parallel to the coordinate axes of R^n, with Δ a proper subset of Δ_1 and $f_1\in H_p^{(r)}(\Delta_1, M_1)$ for a suitable M_1. If, in addition, $m=(\mu_1, \cdots, \mu_n)$ with μ_i $(i=1, 2, \cdots, n)$ a non-negative integer and $\sum \mu_i/\rho_i<1$, then, setting $r'=r(1-\sum \mu_i/\rho_i)$, $f_1^{(m)}\in H_p^{(r')}(\Delta_1, M_1)$ and $M_1+\|f_1^{(m)}\|_{L_p(\Delta_1)}\leq c(M+\|f\|_{L_p(\Delta)})$ for an appropriate c. These results concerning parallelopipeds for the case $p=\infty$ were treated by Hestenes [Duke Math. J. **8** (1941), 183–192; MR **2**, 219] and Whitney [Trans. Amer. Math. Soc. **36** (1934), 63–89; **40** (1936), 309–317].

A. E. Livingston (Seattle, Wash.).

Friedman, A. On the mean value theorem. Bull. Res. Council Israel. Sect. A. **6** (1956), 47–49.

If $\int_{K_R} u(x)dx\leq -\varepsilon(\varepsilon>0)$, K_R is a sphere of radius R and $u(x)$ is continuous, then $u(x)$ takes negative values in a concentric sphere $K_{\theta R}$, $0<\theta<1$. It is shown that for some families of functions there exist θ which depend only on ε and the family. *Author's summary.*

Sawyer, W. W. On determinants associated with Hilbert's inequality. J. London Math. Soc. **32** (1957), 133–138.

It is well known that the integral

$$E=\int_1^n\int_1^n (x+y)^{-1}f(x)f(y)dxdy\Big/\int_1^n f^2(x)dx$$

is, for every real function $f(x)$, less than π, and that its value can be made as near as we please to π by putting $f(x)=x^{-\frac{1}{2}}$ and taking n sufficiently large. The author is concerned with certain forms of $f(x)$ which maximise E for a fixed value of n; he proves that if E is transformed to

$$E=2\iint (u^2+v^2)^{-1}\phi(u)\phi(v)dudv\Big/\int u^{-1}\phi^2(u)du$$

(where the integrals, both in u and in v, are over the interval $(n^{-\frac{1}{2}}, 1)$), and $\phi(u)$ is assumed to be a polynomial of degree N, then the maximum value of E is attained when $\phi(u)$ has the form of a certain determinant, and that this maximum is $\pi-2\sigma_N\nu^{-1}+O(\nu^{-2})$, where $\nu=\frac{1}{4}\log n$ and σ_N can also be expressed as a determinant.

U. S. Haslam-Jones (Oxford).

★ **Apostol, Tom M. Mathematical analysis: a modern approach to advanced calculus.** Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957. xii+553 pp. $8.50.

This is a book on advanced calculus, in that many topics frequently omitted are treated here in detail. On the other hand, the book is easily adaptable to more modest needs. Table of contents: 1. The real and complex number systems. 2. Some basic notions of set theory. 3. Elements of point set theory. 4. The limit concept and continuity. 5. Differentiation of functions of one real variable. 6. Differentiation of functions of several variables. 7. Applications of partial differentiation. 8. Functions of bounded variation, rectifiable curves and connected sets. 9. Theory of Riemann-Stieltjes integration. 10. Multiple integrals and line integrals. 11. Vector analysis. 12. Infinite series and infinite products. 13. Sequences of functions. 14. Improper Riemann-Stieltjes integrals. 15. Fourier series and Fourier integrals. 16. Cauchy's theorem and the residue calculus. The wealth of material is not indicated by this outline alone; e.g., Chapter 6 includes a careful discussion of the differential, and Chapter 9 contains a complete proof for Lebesgue's necessary and sufficient condition for Riemann integrability. Lists of references to other texts accompany most chapters. There are nearly 500 problems, which include counterexamples, and generalizations of results, as well as straightforward exercises. The presentation is simple and clear. An outstanding feature is the unusually careful motivation provided for each new concept (but

with no trace of condescension); likewise, well-chosen comments elucidate the wherefore of proofs. Occasional diagrams, such as to illustrate a suitable smaller neighborhood within a given one, are another welcome feature.
L. Gillman (Lafayette, Ind.).

See also: General Theory of Numbers: Širšov. **Measure, Integration:** Lipiński; Sindalovskiĭ. **Functions of Complex Variables:** Denjoy; Levin. **Harmonic Functions, Convex Functions:** Redheffer; Kurepa. **Approximations, Orthogonal Functions:** Paszkowski. **Trigonometric Series and Integrals:** Baiada. **Differential Geometry:** Haupt. **Numerical Methods:** Ottaviani. **Probability:** Rényi.

Measure, Integration

Lipiński, J. S. Sur la dérivée d'une fonction de sauts. Colloq. Math. **4** (1957), 197–205.

A real-valued, bounded, non-decreasing function f on the real numbers is a non-decreasing jump function if $\sum_{-\infty<x<\infty}[f(x+0)-f(x-0)]<\infty$ and $f(b-0)-f(a+0)=\sum_{a<x<b}[f(x+0)-f(x-0)]$ whenever $a<x<b$. The author proves quite simply, without reference to known theorems concerning the differentiability of monotone functions, that (a) if ϕ is a difference of two non-decreasing jump functions, then $\{x;\ \phi'(x)=\infty\}$ is a subset of an F_σ set of measure zero and (b) if E is an F_σ set of measure zero, then there is a non-decreasing jump function ψ such that $\psi'(x)=\infty$ on E. A consequence of (a) and (b) is that a necessary and sufficient condition for there to be a non-decreasing jump function with infinite derivative on E is that E be a first category set of measure zero; this answers a question posed by E. Marczewski [Prace Mat. 1 (1955), 141–144; MR 17, 136]. *A. E. Livingston.*

Sindalovskiĭ, G. H. On certain questions of continuity of measurable functions. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 175–182. (Russian)

Let ϕ and f be real-valued functions on [0, 1] with f Lebesgue-measurable there and $\phi(0)=0$. Set

$$\Delta_2^\phi(x,h)=f(x-\phi(h)+h)-2f[x-\phi(h)]+f(x-\phi(h)-h).$$

The author proves: (a) If $\phi\in\mathrm{Lip}\,1$ on [0, 1] and $\lim_{h\to0}\Delta_2^\phi f(x,h)=0$ on an x set E of positive measure, then f is continuous for a.a. x in E; (b) if $\lim_{h\to0}\Delta_2^\phi f(x,h)=0$ uniformly on closed subintervals of (0, 1), then $f\in C$ on (0, 1). He points out that a corollary to his proof of (b) is that if f is non-measurable on an interval on which $\Delta_2^\phi(x,h)\to0$ uniformly, then the points of discontinuity of f are nowhere dense in that interval. With reference to (a), he remarks that he has constructed an everywhere discontinuous measurable function F which satisfies $\lim_{h\to0}\Delta_2^\theta F(x,h)=0$ ($\theta(h)=0$ for $0\leqq h\leqq1$) on a perfect x set of measure zero.
A. E. Livingston (Seattle, Wash.).

Takahashi, Shigeru. On some random Riemann-sums. Tôhoku Math. J. (2) **8** (1956), 245–257.

Let $t_{1,n}\leqq t_{2,n}\leqq\cdots\leqq t_{n,n}$ denote the first n ordered values of a sequence of independent random variables having uniform distribution on [0, 1], and put $t_{n+1,n}=1$. It is shown that if $f\in L_p(0,1)$, $p>1$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(t_{i,n})(t_{i+1,n}-t_{i,n})=\int_0^1 f(t)\,dt$$

with probability one. For $f\in L_1(0,1)$ only convergence in probability is proved. [For related results by the author see Proc. Japan Acad. **31** (1955), 8–13; MR **16**, 1009.]
W. Hoeffding (Chapel Hill, N.C.).

Enomoto, Shizu. Sur une totalisation dans les espaces de plusieurs dimensions. II. Osaka Math. J. **7** (1955), 157–178.

Let f be a real function defined on an n-dimensional interval R_0. In the first part of this paper [same J. **7** (1955), 69–102; MR **17**, 246] the author showed that, for $n=1$, f is $\mathscr{D}_*$-integrable on R_0 (i.e. Denjoy-integrable in the restricted sense) if and only if its indefinite $\mathscr{D}_*$-integral has a certain approximation property relating it to the indefinite Lebesgue integral of f. For arbitrary n, he defined f to be $(\mathscr{D})$-integrable on R_0 if there is a finitely additive real function of the interval $I\subset R_0$, called indefinite $(\mathscr{D})$-integral of f, which has a similar approximation property. In the present second part he establishes several properties of this definition, including the following ones: 1) If f depends on one coordinate only as function of which it is $\mathscr{D}_*$-integrable, then it is $(\mathscr{D})$-integrable. 2) If f is $(\mathscr{D})$-integrable on R_0, then it is equal to the ordinary derivative of its indefinite $(\mathscr{D})$-integral at almost all points of R_0. 3) If the strong derivative of a finitely additive real function F of the interval $I\subset R_0$ exists at all points of R_0, then it is $(\mathscr{D})$-integrable on R_0 and F is its indefinite $(\mathscr{D})$-integral. *H. M. Schaerf.*

Choquet, Gustave. Existence des représentations intégrales dans les cônes convexes. C. R. Acad. Sci. Paris **243** (1956), 736–737.

As in the preceding two notes of this series [same C. R. **243** (1956), 555–557, 699–702; MR **18**, 219, 288] B is a compact base of a cone C, and E is the set of extreme points of B. In Note 1, it was shown that if C is a lattice then every point of C is the resultant of at most one non-negative Radon measure carried by E. In Note 2, it was shown that if B is metrizable then every point of C is the resultant of at least one such measure. The first part of this note states conditions for which the latter result holds even if B is not metrizable. In the second part of the present note, C is assumed to be a lattice. Then, if B is a compact metrizable simplex, it follows from results of Notes 1 and 2 that every point of B is the barycenter of one and only one non-negative Radon measure carried by E. This has applications to important problems in classical analysis. Various new concepts and theorems (not mentioned here) are introduced by the author in carrying out the program of the three notes, and it is to be hoped that he plans to publish a detailed development.
C. Goffman (Lafayette, Ind.).

Prékopa, A. On additive and multiplicative totals. Acta Math. Acad. Sci. Hungar. **8** (1957), 107–126.

Betrachtet werden Funktionen f, die auf einem Halbring $\mathscr{X}$ von Teilmengen eines kompakten metrischen Raumes definiert sind und deren Werte in einer Banachschen Algebra $\mathscr{B}$ mit einer Einheit e liegen. Durch eine „Permutationsfunktion" $\mathscr{P}$ in $\mathscr{X}$ werden die Komponenten jeder in $\mathscr{X}$ enthaltenen endlichen disjunkten Zerlegung einer beliebigen Menge aus $\mathscr{X}$ so angeordnet, daß bei einer Verfeinerung $(B_1, B_2, \cdots)$ einer Zerlegung $(A_1, A_2, \cdots)$ aus $B\subseteq A_k$, $B_j\subseteq A_i$ und $k<l$ folgt $i<j$. Die Funktion f heißt multiplikativ, wenn es eine Permutationsfunktion $\mathscr{P}$ gibt, so daß für jede gemäß $\mathscr{P}$ angeordnete Zerlegung $(A_1, \cdots, A_r)$ von A gilt $f(A)=\prod_i f(A_i)$; jedoch wird bei kommutativem $\mathscr{B}$ die Existenz von $\mathscr{P}$

nicht vorausgesetzt. Die Definition der Additivität von f ist die übliche. Durch Verallgemeinerung des Burkillschen Integrationsprozesses wird das additive Total von f über eine Menge B aus $\mathscr{X}$ mit Hilfe der Summen $\sum_i f(A_i)$, in denen $(A_1, \cdots, A_r)$ die Zerlegungen von B durchläuft, definiert, und analog das multiplikative Total von f über B mit Hilfe der Produkte $\prod_i f(A_i)$, wobei $(A_1, \cdots, A_r)$ durch $\mathscr{P}$ angeordnet sei. Unter geeigneten Voraussetzungen wird die Existenz des additiven Totals von $f-e$ bewiesen, wenn f multiplikativ und $f-e$ von beschränkter Variation ist, und entsprechend die Existenz des multiplikativen Totals von $f+e$, wenn f additiv und von beschränkter Variation ist. Die beiden Prozesse sind unter diesen Voraussetzungen invers: ist zum Beispiel g additiv und f das multiplikative Total von $g+e$, so wird g das additive Total von $f-e$. Der Verf. weist auf analoge Sätze von Dobrušin [Mat. Sb. N.S. **33(75)** (1953), 567–596; MR **15**, 542] hin, die sich auf den Fall beziehen, daß $\mathscr{A}$ eine Matrizenalgebra ist; außerdem bezogen sich alle früheren Theorien multiplikativer Totale (Integrale) nur auf den Fall eines Semirings aus Intervallen der Zahlengeraden, in dem die natürliche Anordnung der Intervalle eine Permutationsfunktion darstellt. Zwei Beispiele, aus der Theorie der stochastischen Mengenfunktionen und der Markovschen Prozesse, werden behandelt.

K. Krickeberg (Würzburg).

Fréchet, Maurice. Les inégalités de Minkowski dégénérées et leurs applications en calcul des probabilités. Ann. Inst. H. Poincaré **15** (1956), 1–33.

Let (X, Y) denote the distance of random variables taking their values in a metric space. The author proved [same Ann. **10** (1948), 215–310; MR **10**, 311] that $d_\alpha(X, Y)$, $=E(X, Y)^\alpha$ for $\alpha<1$ and $=(E(X, Y)^\alpha)^{1/\alpha}$ for $\alpha\geq 1$, obeys the triangle inequality. He shows that this expression has limits $\alpha\to 0$ and as $\alpha\to\infty$. In the first case, if $P[X=Y]\neq 0$, the as limit is gen. max. $(X, Y)=\sup\rho$ over all $\rho>0$ such that $P[(X, Y)>\rho]=0$. In the second case the limit is $P[X\neq Y]$. Necessary and sufficient conditions for the triangle inequality to become equality are obtained for $0\leq\alpha\leq\infty$; algebraic and analytical applications are given. Stochastic convergences compatible with convergence in distance are considered. Families of functions g are introduced such that convergence in probability is compatible with convergence in distance $d(X, Y)=Eg(X, Y)$; this generalizes the author's results in his classical book [Généralités sur les probabilités, éléments aléatoires, 2ième éd., t. 1, fasc. 3, livre 1, Gauthier-Villars, Paris, 1950; MR **12**, 423]. In addition, it is shown that convergence in distance gen. max. (X, Y) is equivalent to uniform almost sure convergence. *M. Loève* (Berkeley, Calif.).

See also: Special Functions: Lorch and Szego. **Topological Groups:** Pettis. **Topological Vector Spaces:** Lowdenslager, Łojasiewicz (two articles). **Banach Spaces, Banach Algebras, Hilbert Spaces:** Hewitt and Zuckerman.

Functions of Complex Variables

Morev, I. A. Monogenic hypercomplex functions. Ukrain. Mat. Ž. **8** (1956), 423–434. (Russian)

Let A be an associative commutative finite-dimensional algebra with unit, over the complex numbers. Let $e_1, \cdots, e_m$ be a basis for A and consider $\zeta=\sum a^k e_k$, $f=\sum b^k e_k$, $\psi=\sum c^k e_k$, where a^k, b^k, c^k are complex functions over a region D in Euclidean n-space. The author calls f monogenic with respect to ζ, and writes $f=f\{\zeta; D\}$, if there is a ψ such that each component of $\Delta f-\psi(M)\cdot\Delta\zeta$ is $o(|MM'|)$ as $M'\to M(\Delta f=f(M')-f(M))$, etc.); he calls ψ the first derivative of f with respect to ζ. There is a similar definition for functions monogenic with respect to several hypercomplex variables $\zeta^1, \cdots, \zeta^p$, $p>1$. The author investigates properties of such functions, including an integral theorem, stating that previous work has been restricted to $p=1$, and [except for Fedorov, Mat. Sb. N.S. **18(60)** (1946), 353–378; MR **8**, 25] to functions of a more special kind. His principal aim is to obtain conditions on $\zeta^1, \cdots, \zeta^p$ under which every f which is monogenic with respect to these variables in D has as its components b^k harmonic point functions in D, and also has derivatives of all orders with respect to $\zeta^1, \cdots, \zeta^p$ which are themselves monogenic in D with respect to $\zeta^1, \cdots, \zeta^p$.

He writes $f=^*f\{\zeta; D\}$ if $f=f\{\zeta; D\}$ and $\partial\zeta/\partial x^1$ and $(\partial\zeta/\partial x^1)^{-1}$ exist in D; and $f=^{**}f\{\zeta; D\}$ if in addition $n>1$ and $\zeta=ax^1+\theta(x^2, \cdots, x^n)$, where a is an element of A and θ is a hypercomplex function each of whose components has continuous derivatives of all orders with respect to $x^2, \cdots, x^n$; there are similar definitions for $p>1$. The final two theorems are as follows. Theorem 3: Let $\mathfrak{M}$ consist of all those hypercomplex functions f in a given region D such that $f=^{**}f\{\zeta^1, \cdots, \zeta^p; D\}$ for given $\zeta^1, \cdots, \zeta^p$ (the same for all elements of $\mathfrak{M}$). Then in order that every element of $\mathfrak{M}$ is harmonic with respect to $x^1, \cdots, x^n$ it is necessary and sufficient that $\nabla^2\zeta^j=0$ and $\sum_{i=1}^n \zeta_i^j\zeta_i^h=0$ for all j, h ($j, h=1, \cdots, p$; $\zeta_i^j=\partial\zeta^j/\partial x^i$). Theorem 4: If $f=^*f\{\zeta^1, \cdots, \zeta^p; D\}$ and $f, \zeta^1, \cdots, \zeta^p$ have continuous derivatives of all orders with respect to $x^1, \cdots, x^n$ then f has derivatives of all orders with respect to $\zeta^1, \cdots, \zeta^p$. *E. S. Pondiczery* (Princeton, N.J.).

Perry, R. L. The zeros of a family of integral functions. J. London Math. Soc. **32** (1957), 144–153.

Let $g(z)=1+\sum_{n=1}^\infty a_n z^n$ be an entire function, not a polynomial, let $r_n(t)$ be the Rademacher functions, and let $f(z, t)=\sum_{n=0}^\infty r_n(t)a_n z^n$. Let S be the set of points which are zeros of some $f(z, t)$. The author shows that S is perfect and of zero Hausdorff dimension. He states that if $r_n(t)$ are replaced by a set of independent functions $\alpha_n(t)$ the same results hold provided that $\alpha_n(t)$ takes only a finite set of nonzero values; but if $\alpha_n(t)$ has range dense in some interval, S has dimension 2. *R. P. Boas, Jr.*

Specht, Wilhelm. Die Lage der Nullstellen eines Polynoms. Math. Nachr. **15** (1956), 353–374.

Let the polynomials in the complex variable z, $p_n(z)=p_{n0}+p_{n1}z+p_{nn}z^n$ ($n=0, 1, \cdots$), form an orthonormal set relative to $g(\theta)$, a non-negative, Lebesgue-integrable weight function with period 2π in the real variable θ; that is, $(\frac{1}{2}\pi)\int_{-\pi}^{\pi} p_m(e^{i\theta})\overline{p_n(e^{i\theta})}g(\theta)=\delta_{mn}$. Any polynomial $P(z)$ of degree n may then be written as a linear combination of the $p_n(z)$: $P(z)=b_0p_0(z)+b_1p_1(z)+\cdots+b_np_n(z)$. The relation between the values of the $b_0, b_1, \cdots, b_n$ and the position of the zeros of $P(z)$ is investigated. Among the theorems that are proved with the aid of known properties of orthogonal polynomials is that all the zeros of $P(z)$ lie in the disk $|z|\leq B$, where $|b_n|^2B^2=|b_0|^2+|b_1|^2+\cdots+|b_n|^2$. If $h(X)$ is a real-valued non-negative function of a real variable x for which $x^nh(x)$ is Lebesgue-integrable for $-\infty<x<\infty$, and if the $p_n(z)$ are defined by the equations $\int_{-\infty}^{\infty} p_m(x)p_n(x)h(x)dx=\delta_{mn}$, then the zeros are located in various strips. For example,

every zero $Z=X+iY$ of $P(z)$ is shown to satisfy the relation $|Y|\leqq kC$, where k is a certain constant independent of $P(z)$ and $C^2=B^2-1$. *M. Marden.*

Parodi, Maurice. Sur les polynomes ayant un zéro de module supérieur à l'unité. C. R. Acad. Sci. Paris **244** (1957), 1436–1437.

Given the polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n \quad (a_n\neq 0)$$

in which $|a_1|>1+b$ and $b=\sum_{k=2}^{n}|a_k|$, the author [same C. R. **242** (1956), 598–600; MR **17**, 722] concluded from the assumption $b>1$ that $n-1$ zeros of $f(z)$ lie in the unit circle and the one outside this circle lies in the disk $|z+a_1|\leqq 1$. In the present paper he shows that the same result may be deduced without any assumption on b. *M. Marden* (Milwaukee, Wis.).

Parodi, Maurice. Application de la méthode de Chiò de développement d'un déterminant, à la localisation des zéros d'un polynome. C. R. Acad. Sci. Paris **244** (1957), 2269–2270.

The polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$, $a_n\neq 0$, may be written as the determinants

$$f(z)=(-1)^n\begin{vmatrix} -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -(a_1+z) \end{vmatrix}$$

$$=(-1)^{n+1}\times\begin{vmatrix} -z^2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -(a_n+a_{n-1}z) & -a_{n-2} & -a_{n-3} & \cdots & -a_2 & -(a_1+z) \end{vmatrix}$$

When applied to the latter determinant, the Hadamard theorem shows that the zeros of $f(z)$ lie in the union of the interiors of the circle $|z|\leqq 1$ and the Cartesian oval $|z+a_1|-|a_n+a_{n-1}z|\leqq|a_2|+|a_3|+\cdots+|a_{n-2}|$. *M. Marden* (Milwaukee, Wis.).

Abe, Hitoshi. A note on subordination. J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1956), 47–51.

Some elementary modifications of the classical applications of the elliptic modular function to function-theoretic problems. *Z. Nehari* (Pittsburgh, Pa.).

Heinhold, J. Zur konformen Abbildung schlichter Gebiete. Math. Z. **67** (1957), 133–138.

The notion of Koebe's osculating function ("Schmiegungsfunktion") is generalized by taking a reference functional into consideration as the measure of approximation. A general convergence theorem is proved in establishing the existence of a mapping onto a normal domain. It is then applied especially to obtain an iteration process for schlicht mappings of a ring domain onto an annulus. *Y. Komatu* (Tokyo).

Bieberbach, Ludwig. Eine Bemerkung zur konformen Abbildung zweifach zusammenhängender Gebiete. Math. Z. **67** (1957), 99–102.

In relation to the mapping of a multiply-connected domain onto a many-sheeted circular disc, the author proves the following theorem. Let C_1 and C_2 be boundaries of two non-disjoint simply-connected schlicht domains G_1 and G_2, respectively. Let further C_1 and C_2 be both continua and have at least one common point which is accessible from $G_1\cap G_2$ along a common path. Then any ring domain can be mapped one-to-one and conformally onto a two-sheeted domain which is bounded by a component lying on C_1 in one sheet and by a component lying on C_2 in another sheet, and which is constructed by connecting G_1 and G_2 crosswise along a cut between two suitably chosen branch points in $G_1\cap G_2$. *Y. Komatu.*

Kuvaev, M. P.; and Kufarev, P. P. An equation of Löwner's type for multiply connected regions. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 19–34. (Russian)

Let B_0 be a $(p+1)$-tuply connected region of the z-plane which contains the point $z=0$ and is bounded by $p+1$ closed curves $\Gamma_0, \Gamma_1, \cdots, \Gamma_p$ whose spherical images are Jordan curves. Let C be a Jordan arc with parametric equation $z=\varphi(s)$, $t_0\leqq s\leqq t_1$, lying wholly in B_0, with the exception of the point $z=\varphi(t_0)$ which belongs to Γ_0, and let $C(t)$ denote that subarc of C which results by restricting the parameter s to the interval $t_0\leqq s\leqq t$, where $t\leqq t_1$. Denoting by $B(t)$ the region obtained from B_0 by slitting it along the arc $C(t)$, let $z=\Phi(w, t)$ be the regular function in $|w|<1$ which maps $|w|<1$ conformally onto the universal covering surface of $B(t)$, so that $\Phi(0, t)=0$, $\partial\Phi(0, t)/\partial w>0$. The purpose of this paper is to derive a differential equation for $\Phi(w, t)$ which is analogous to the classical Loewner differential equation in the case of simply connected regions. [In this connection, cf. also Y. Komatu, Proc. Japan Acad. **26** (1950), no. 7, 26–31; MR **13**, 734.] *W. Seidel* (Notre Dame, Ind.).

Tsuji, Masatsugu. On the moduli of closed Riemann surfaces. Comment. Math. Univ. St. Paul. **5** (1956), 25–28.

If two Riemann surfaces F and F' are conformally equivalent, then F and F' are said to belong to the same class. Different classes of closed Riemann surfaces of the same genus $p\geqq 1$ form a manifold M, such that a point of M and a class correspond in a one-to-one manner. Let μ be the dimension of M in the real sense; then M depends on μ real parameters which are called moduli of closed Riemann surfaces of genus $p\geqq 1$. Denoting a unit circle by C, a straight line by L, a half line by L_0, and the product space $C\times\cdots\times C$ (p times) by C^p, etc., the author proves the following. If $p=1$, then $\mu=2$ and $M=C\times L_0$. If $p\geqq 2$, then $\mu=6p-6$, and further if $\mathfrak{M}=C^p\times L_0^p\times L^{4p-6}$, then $\mathfrak{M}$ is decomposed by singular manifolds into a countable number of connected manifolds $\mathfrak{M}_i$, such that each $\mathfrak{M}_i$ can be taken as M. *K. Noshiro* (Nagoya).

Denjoy, Arnaud. La fonction minkowskienne complexe uniformisée éclaire la genèse des fractions continues canoniques réelles. C. R. Acad. Sci. Paris **242** (1956), 1817–1823.

A system of cuts of the function $\chi(z, \alpha)$, and the path that it imposes on a complex number z tending toward a real number x, gives the sequence of approximants of the development of x into a continued fraction. *E. Frank.*

Denjoy, Arnaud. La fonction minkowskienne complexe uniformisée détermine les intervalles de validité des transformations de la fonction réelle. C. R. Acad. Sci. Paris **242** (1956), 1924–1930.

Through the discontinuities of the complex function $\chi(z, \alpha)$ in the opening of uniform cuts, one explains the changes under linear transformations of the real function $\chi(x, \alpha)$ corresponding to a like substitution of the Schwarz group. *E. Frank* (Chicago, Ill.).

Denjoy, Arnaud. Propriétés différentielles de la fonction minkowskienne réelle. Statistique des fractions continues. C. R. Acad. Sci. Paris **242** (1956), 2075–2079.

It is shown that the real, continuous, decreasing Minkowski function $\chi(x, \alpha)$ has not at any point a finite non-zero derivative. If two extreme derivatives are finite and negative, their ratio surpasses 17/16. When the n first partial quotients of a normal continued fraction are ranked in order of decreasing size, one can find a bound for the qth. *E. Frank* (Chicago, Ill.).

Maude, Ronald. Exceptional sets with respect to order of integral functions of two variables. Proc. Cambridge Philos. Soc. **53** (1957), 323–342.

The total order of an entire function $f(z_1, \cdots, z_n)$ of n complex variables is the inf of all positive numbers p such that $|f(z_1, \cdots, z_n)| < \exp(r^p)$ whenever $|z_1|^2 + \cdots + |z_n|^2 = r^2$ and r is large enough. Let ρ be the total order of $f(z, w)$; for every complex number t let f_t be defined by $f_t(z) = f(z, tz)$; and let $\rho(t)$ be the order of f_t. Lelong [Ann. Sci. Ecole Norm. Sup. (3) **58** (1941), 83–177; MR **7**, 151] has shown that $\rho(t) \leqq \rho$ for all t and that the exceptional set (i.e., the set of all t for which $\rho(t) < \rho$) has exterior capacity zero. The author describes the exceptional set more explicitly in terms of the zeros of the polynomials $P_n(t)$ which are generated by the expansion $f_t(z) = \sum_0^\infty P_n(t) z^n$. From this description he deduces the fact that every exceptional set of the above sort is an F_σ whose Hausdorff logarithmic dimension does not exceed 1; conversely, every F_σ whose Hausdorff logarithmic dimension is less than 1 is an exceptional set for some $f(z, w)$ of finite total order. The proof of the converse depends on a chain of intricate lemmas concerning F_σ sets, of which we state the simplest: Every F_σ in a metric space is the union of a countable collection of closed bounded sets F_i, such that no F_i intersects more than two others and such that no three F_i's intersect. *W. Rudin* (Rochester, N.Y.).

★ **Левин, Б. Я. [Levin, B. Ya.] Распределение корней целых функций. [Distribution of zeros of entire functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 632 pp. 17.30 rubles.

This is the third book on entire functions to have appeared in the last few years; its objectives and its contents differ considerably from those of the other two [Boas, Entire functions, Academic Press, New York, 1954; Cartwright, Integral functions, Cambridge, 1956; MR **16**, 914; **17**, 1067], although a certain amount of overlap is inevitable. The first half of the book consists mainly of material that is fairly well covered, or is even covered in more detail, in other books or in journals that will be more accessible to Western readers; the second half consists mainly of material that has not previously appeared in books in any language, except that there is some overlap with Čebotarev and Meiman's monograph [The Routh-Hurwitz problem..., Trudy Mat. Inst. Steklov. **26** (1949); MR **11**, 509]. A number of the author's own results, previously announced without proof or only with sketches of proofs, are proved in detail. The book will be a useful reference for much interesting material that is not easily accessible elsewhere.

The first chapter contains general results, stated wherever possible in terms of proximate orders. The second and third chapters deal with the connections between over-all regularities of distribution of zeros and regularity of growth. If the zeros are so regularly distributed that they have an appropriate kind of density in every angle, the function behaves in a correspondingly regular way; and conversely. The direct results were obtained by the author and independently by Pfluger, who also found the converse results; the account in Pfluger's papers is more detailed and will be more accessible to Western readers [Comment. Math. Helv. **11** (1938), 180–213; **12** (1939), 25–65; **14** (1942), 314–349; **18** (1946), 177–203; MR **1**, 113; **5**, 258; **7**, 426].

Chapter 4 is entitled "Uniqueness, interpolation and completeness", the general theme being that a function cannot have too many zeros for its rate of growth without vanishing identically. Chapter 5, "Functions of class A", is concerned with entire functions, or functions analytic in the upper half plane, whose zeros a_k satisfy $\sum |\Im(1/a_k)| < \infty$, so that the zeros are "nearly" real; and especially with functions which are both of exponential type and of class A. A typical theorem connects the growth of the function along the real axis with its growth along rays from the origin. There is a somewhat more detailed discussion in Boas's book.

Chapter 6 is called "Zeros of exponential sums". When an entire function of exponential type is almost periodic, very precise statements can be made about the distribution of its zeros and its "mean motion". These form the first half of the chapter. The second half deals with the rate of growth and the zeros of entire functions which can be approximated by exponential sums whose exponents belong to a given set. When the set is a polygonal curve the functions form a natural generalization of the almost periodic functions considered in the first part of the chapter.

Chapter 7 is on the Hermite-Biehler theorem for entire functions. The original Hermite-Biehler theorem says that the polynomial $P+iQ$ has no zeros in $y \leqq 0$ if and only if P and Q have simple interlacing real zeros, and satisfy $Q'P - QP' > 0$ for real x. Considerable effort has been devoted to finding classes of entire functions for which the theorem remains true; the subject is also surveyed in Čebotarev and Meiman's monograph. Some of the results are as follows. An entire function $\omega(z)$ is said to belong to HB ($\overline{HB}$) if it has no zeros in $y \leqq 0$ and $|\omega(z)/\bar{\omega}(z)| < 1 (\leqq 1)$ for $y > 0$; the second condition is automatic for polynomials. An entire function of exponential type, $\omega(z) = P(z) + iQ(z)$, belongs to HB if it has zeros, all in $y \geqq 0$, and the zeros of P and Q separate each other. The function ω is said to belong to P (or $\bar{P}$) if it has no zeros in $y < 0$ (or $y \leqq 0$) and its indicator satisfies $h(-\pi/2) \geqq h(\pi/2)$, so that it grows faster in the zero-free half-plane than in the other half-plane. For functions of exponential type, classes $\overline{HB}$ and P coincide.

Chapter 8 deals with approximation of entire functions by polynomials whose zeros are in a given region. Most of the results are classical; the author has apparently overlooked Korevaar's more general work [Duke Math. J. **18** (1951), 573–592; MR **13**, 222]. Chapter 9 deals with operators which preserve inequalities between entire functions. The motivating idea is that S. Bernstein's inequality for the derivative of an entire function of exponential type σ can be written: $|f(x)| \leqq |e^{i\sigma x}|$ implies $|f'(x)| \leqq |(e^{i\sigma x})'|$, and the main objective is to generalize both the operation and the function that appears on the right. A $\mathfrak{B}$-operator is a linear operator which leaves the class $\bar{P}$ invariant; differentiation is a $\mathfrak{B}$-operator. Provided that $\omega(z)$ is of class P and exponential type σ, and $f(z)$ is of exponential type $\tau \leqq \sigma$, the inequality $|f(x)| \leqq$

$|\omega(x)|$ implies $|\mathfrak{B}f(x)| \leq |\mathfrak{B}\omega(x)|$. Other extremal problems are solved and the general form of $\mathfrak{B}$-operators is determined.

The remainder of the book consists of a variety of complements to the general theory and applications. The Paley-Wiener theorem on the representation of a function of exponential type which belongs to L^2 is proved and a generalization is given: if the conjugate indicator diagram of f is a polygon and $f(re^{i\theta})e^{-h(\theta)r}$ belongs to L^2 in each direction θ perpendicular to a side of the polygon, then $f(z)=\int e^{iwz}\phi(w)dw$ with integration on the boundary of the conjugate indicator diagram and $\phi \in L^2$. Further topics include: Quasianalytic classes of almost periodic functions; Completeness of $\{e^{i\lambda_n x}\}$; Application of uniqueness theorems to problems in differential equations (completeness in a region of the solutions of a Sturm-Liouville problem; determination of the equation by its spectrum); Representation of a nonnegative function as a square; More material on almost-periodic functions with bounded spectrum. *R. P. Boas, Jr.* (Evanston, Ill.).

Edrei, A.; and MacLane, G. R. On the zeros of the derivatives of an entire function. Proc. Amer. Math. Soc. **8** (1957), 702–706.

By L_f is meant the point set on the sphere comprised of points z of which every neighborhood contains zeros of infinitely many of the derivatives $f^{(n)}(z)$ of the entire function $f(z)$. Such a set L_f is always compact; if not empty, it contains ∞, as may be concluded from a result of Pólya and Saxer [W. Saxer, Math. Z. **17** (1923), 206–227]. The principal theorem of this paper is the following. If K is any compact set of the complex sphere containing ∞ and if $p(r)$ is a continuous function for $r \geqq 0$ such that $p(0)>0$, $p(r) \uparrow \infty$ and $r^{-c}p(r) \to \infty$ as $r \to \infty$ for every constant c, then there exists an entire function f such that $L_f=K$ and the maximum modulus $M(r)$ of f satisfies the relation $\limsup [M(r)/p(r)]=1$ as $r \to \infty$. *M. Marden.*

Kövári, T. A note on entire functions. Acta Math. Acad. Sci. Hungar. **8** (1957), 87–90.

Let f be entire, of order $\rho<\infty$, type $\sigma<\infty$ ($\rho\sigma \neq 0$) and let $M(r)=\max_{|z|\leq r}|f(z)|$, $M_1(r)=\max_{|z|\leq r}|f'(z)|$. It is known that $M'(r)$ exists, except possibly for a sequence of values of r, and that $M'(r) \leq M_1(r)$. The example $f(z)=\exp(\sigma z^\rho)$ suggests that M' and M_1 might be expected to be of magnitude $M(r)\rho\sigma r^{\rho-1}$. The author shows that for any function f,

$$1 \leq \limsup \frac{M'(r)}{M(r)\rho\sigma r^{\rho-1}} \leq \limsup \frac{M_1(r)}{M(r)\rho\sigma r^{\rho-1}} \leq e,$$

and that 1 and e are best possible. He asserts that in his thesis he has shown that if $f(z)=\sum a_n z^{\lambda_n}$ with $\lambda_{n+1}/\lambda_n \uparrow \infty$, then equality holds for the two inequality signs on the right. *R. C. Buck* (Madison, Wis.).

Crum, M. M. Note on functions of exponential type in a half-plane. Quart. J. Math., Oxford Ser. (2) **6** (1955), 283–287.

If $f(z)$ is regular for $x \geqq 0$ and $\log|f(z)| \leq cx$, and if for a set of points $\zeta_n=\xi_n+i\eta_n$ we have $\log|f(\zeta_n)| \leq a\xi_n$, then $\log|f(z)| \leq ax$, provided that the ζ_n satisfy $\sum \xi_n^2/|\zeta_n|^3<\infty$ and the number of ζ_n such that $|\zeta_n-z|<t$ does not exceed $1+Bt$ with B independent of z; the effect of the last condition is to keep the ζ_n from clustering around the zeros of f. Results of this kind would follow from the Ahlfors-Heins theorem [cf. Hayman, J. Math. Pures Appl. (9) **35** (1956), 115–126; MR **17**, 1073] if the ζ_n formed a considerably denser set. The proof is based on Nevanlinna's representation theorem for $f(z)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Rabinovič, Yu. L. On entire functions representable as Laplace integrals. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 199–221. (Russian)

If $f(z)=\sum_{n=0}^{\infty} c_n z^n$ is an entire function with $|f(z)|<A\exp\{a|z|^\alpha\}$ (A, a, and α positive), elementary inequalities yield $|c_n|<B(a\alpha)^{n/\alpha}/[(n-1)!]^{1/\alpha}$ (B constant). The author then investigates the growth of functions $\Phi(x,\lambda)=\sum_{n=0}^{\infty} x^n/(n!)^\lambda$ (x, λ positive) for $x \to \infty$ and finds finally that the entire function $\Phi(z,\lambda)$ is of order $1/\lambda$ and type $\geqq\lambda$. If λ is rational the type is exactly λ. In a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **60** (1948), 969–972; MR **10**, 110] the author studied the Laplace transform and its inverse of a function $f(z)$ of exponential type (with an isolated singularity at $z=\infty$). He now investigates the Laplace transform of functions (*) $f(z)=z^{\beta-1}\exp\{-az^\alpha\}\psi(z)$ ($\alpha>1$, $a>0$, $\beta>0$) with $\psi(z)$ holomorphic in the angle (A): $(\theta_0-h \leq \arg z \leq \theta_0+h)$ and $|\psi(z)|<B\exp\{b|z|^{\alpha_1}\}$ ($\alpha_1<\alpha$). He finds that if the entire function $F(z)$ is a Laplace transform of (*), then there exists an angular domain (C) with angle $\gamma=(1+1/\alpha)\pi$ in which $|F(z)|<B_1|z|^\beta$. Entire functions with this property are called "of class L_α'''". Further discussions prove that $F(z)$ is of order $\alpha/(\alpha-1)$. § 3 deals with the inverse problem. Let (C) be the angle $0 \leq \arg z \leq \gamma$ ($\gamma=(1+1/\alpha)\pi$, $\alpha>1$) and $F(z)$ of class L_α'. Consider the angle (C_μ): $0 \leq \arg\zeta \leq \mu\gamma$ ($\mu>0$, and $\mu\gamma>\pi$). If $f_\mu=(2\pi i)^{-1}\int \exp\{z\zeta\}F(\zeta^{1/\mu})d\zeta$ (the integral being taken over the boundary of (C_μ)), then $F(z^{1/\mu})=\int \exp\{-z\zeta\}\, f_\mu(\zeta)d\zeta$ (the integral taken from 0 to ∞ along the ray l_θ: $\arg\zeta=\theta$, $\cos\theta<0$, $\cos(\theta+\mu\gamma)<0$). $F(z^{1/\mu})$ is called the transform of $f_\mu(z)$. The behaviour of $f_\mu(z)$ is then studied for $z \to 0$ and $z \to \infty$. Finally a class L_α'' of analytic functions is defined (through a specific behaviour in an angular domain). The class L_α'' is called conjugate to the class L_α' and a one-to-one correspondence (through suitable integral transformations) is established between the functions $F(z)$ of L_α' and the functions $f(z)$ of L_α''. *B. Amirà* (Jerusalem).

Seibert, Peter. Sur les chemins asymptotiques des fonctions méromorphes. C. R. Acad. Sci. Paris **244** (1957), 1443–1445.

Let $f(z)$ be a function meromorphic in the circle $|z|<R \leq \infty$. A curve L (defined by $z=\varphi(t)$, $0 \leq t<1$, where $\varphi(t)$ is a continuous complex function with $\varphi(0)=0$ and $|\varphi(t)|<R$) is called an asymptotic path with the asymptotic value a of the function f, if $\lim_{t\to 1}|\varphi(t)|=R$ and $f(\varphi(t))$ tends to a when $t \to 1$. Using the topology introduced by Mazurkiewicz [Fund. Math. **17** (1931), 26–29] the author defines the notion of a class of asymptotic paths and proves the existence of a meromorphic function for which the space of classes of asymptotic paths has a given structure. *B. Amirà* (Jerusalem).

Lehto, Olli; and Virtanen, K. I. Boundary behaviour and normal meromorphic functions. Acta Math. **97** (1957), 47–65.

The authors investigate the properties of a conformally invariant class of meromorphic functions defined in a general domain G. The class of functions, which are called normal in G, is defined as follows. If $f(z)$ is meromorphic in a simply connected domain G, then $f(z)$ is normal in G if and only if the family $\{f(S(z))\}$, where $S(z)$ is an arbitrary one-to-one conformal mapping of G onto itself, is normal

in the sense of Montel. If G is multiply connected, $f(z)$ is normal in G if it is normal on the universal covering surface of G. It is shown first that a non-constant function $f(z)$ is normal in G if and only if there exists a finite constant C such that $\rho(f(z))|dz|=|f'(z)||dz|/[1+|f(z)|^2]\leqq C d\sigma(z)$, where $d\sigma(z)$ is the element of length in the hyperbolic metric of G, so that, for example, the class includes the class of bounded functions, the schlicht functions, and functions omitting at least three values.

The notion of normal meromorphic function arises naturally in the study of the asymptotic values of a meromorphic function: it is shown that if $f(z)$ is normal in G and has an asymptotic value α at a boundary point P of G along an arc lying in $\overline{G}$, then $f(z)$ has α as an angular limit at P. The authors then extend various results of Lindelöf to their class of functions; the following theorem is typical. If $f(z)$ is meromorphic in $|z|<1$ and has a radial limit α at the point $z=e^{i\beta}$, then $f(z)$ has α as an angular limit at $e^{i\beta}$ if and only if $\rho(f(z))=O(1)/(1-|z|)$ in every angle $|\arg(1-ze^{-i\beta})|<\frac{1}{2}\pi-\delta$, $\delta>0$. Although the class of normal functions overlaps the class of functions of bounded charàcteristic, examples are given to show that the two classes are not identical; in particular, the authors give an example of a normal meromorphic function in $|z|<1$ which has a radial limit at no point of $|z|=1$, so that a normal function need not possess any asymptotic values at all.

A generalization of the so-called 'Two Constants' theorem is then proved which yields a sharp form of the Phragmèn-Lindelöf theorem together with the interesting result that a meromorphic function cannot be normal in any neighbourhood of an isolated essential singularity.

A. J. Lohwater (Ann Arbor, Mich.).

Lehto, Olli; and Virtanen, K. I. On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity. Ann. Acad. Sci. Fenn. Ser. A. I. no. **240** (1957), 9 pp.

In a previous paper [cf. preceding review] the authors have shown that, in any neighborhood of an isolated essential singularity, a meromorphic function cannot be normal. In the present paper they consider the problem of characterizing the growth of $f(z)$ near the singularity by means of the spherical derivative $\rho(f(z))=|f'(z)|/(1+|f(z)|^2)$. It is proved first that if $f(z)$ is single-valued and meromorphic in a neighborhood of the isolated essential singularity $z=\infty$, then there exists a universal constant $k>0$ such that $\limsup_{z\to\infty}|z|\rho(f(z))\geqq k$ for all such $f(z)$, while, for arbitrary $\varepsilon>0$, there exist functions for which $\limsup_{z\to\infty}|z|\rho(f(z))<k+\varepsilon$. Even though a meromorphic function cannot be normal in any neighbourhood of the singularity $z=\infty$, the authors are able to characterize the growth of $f(z)$ by introducing the concept of "weak normality": a function $f(z)$, meromorphic in a domain G, is said to be weakly normal in G if it is normal in every simply connected subdomain of G. It is then proved that if $f(z)$ is meromorphic in the neighbourhood V: $0<|z_0|\leqq|z|<\infty$ of the essential singularity $z=\infty$, then $\limsup_{z\to\infty}|z|\rho(f(z))<\infty$ if and only if $f(z)$ is weakly normal in V. An immediate consequence of this last result is that a function weakly normal in V cannot possess any asymptotic values and thus must assume all complex values infinitely often. The authors show finally that their class of weakly normal functions is equivalent to the class of Julia exceptional functions.

A. J. Lohwater (Ann Arbor, Mich.).

★ **Hayman, W. K. The coefficients of schlicht and allied functions.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 102–108. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author gives a brief survey on the coefficients of univalent (schlicht), multivalent, and mean p-valent functions with the principal emphasis on his results on the asymptotic behaviour of such coefficients [C. R. Acad. Sci. Paris **237** (1953), 1624–1625; Proc. London Math. Soc. (3) **5** (1955), 257–284; MR **15**, 516; **17**, 142].

J. A. Jenkins (Princeton, N.J.).

Komatu, Yûsaku. On the coefficients of typically-real Laurent series. Kôdai Math. Sem. Rep. **9** (1957), 42-48.

Let the Laurent series $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ represent in $\rho<|z|<1$ a regular and univalent function $f(z)$ which is real for real z. In a paper by the reviewer and Schwarz [Proc. Amer. Math. Soc. **5** (1954), 212–217; MR **15**, 786] it was shown that the coefficients a_n of such a series are subject to the inequality

$$(*)\qquad (1-\rho^{2n})|a_n|\leqq n|a_1(1+\rho^{n+1})-a_{-1}(1+\rho^{n-1})|.$$

While this inequality is sharp as it stands, the present author notes that, if $f(z)$ is normalized by the condition $\mathrm{Im}\{f(z)\}\mathrm{Im}\{z\}\geqq 0$, (*) yields in the case of odd n either the maximum of a_n or of $-a_n$, but not both. By a refinement of the method of proof, the author obtains the exact upper and lower bounds for a_n in this case.

Z. Nehari (Pittsburgh, Pa.).

Kufarev, P. P. Remark on the problem of coefficients. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 15–18. (Russian)

Let S denote the class of functions $f(w)=w+\sum_{n=2}^{\infty}c_nw^n$, holomorphic and univalent in $|w|<1$. For given $n\geqq 2$, let $f(w)$ be a function in S for which $\Re c_n$ attains its maximum. If the image of $|w|<1$ under the mapping $z=f(w)$ is bounded by rectilinear slits, then $\Re c_n\leqq n$. {The author's proof of this assertion is not clear to the reviewer.}

W. Seidel (Notre Dame, Ind.).

Klingen, Helmut. Über die analytischen Abbildungen verallgemeinerter Einheitskreise auf sich. Math. Ann. **132** (1956), 134–144.

Generalizing a previous result [Math. Ann. **129** (1955), 345–369; MR **18**, 299], the author treats the analytic mappings of the unit circle on itself in a space of several complex variables. Let Z be the $p\times q$ complex coordinate matrix; $\tilde{Z}$, the complex conjugate of the transpose of Z; and $\Omega_{p,q}$, the group of all solutions of $\tilde{M}K_{p,q}M=K_{p,q}$, where $K_{p,q}=\begin{pmatrix}-E^{(p)} & 0\\ 0 & E^{(q)}\end{pmatrix}$ and $E^{(p)}$ is the $p\times p$ unit matrix. The generalized unit circle $\mathfrak{E}_{p,q}$ is defined as the set of points Z for which the hermitian matrix $E^{(q)}-\tilde{Z}Z$ is positive definite. The author proves: the group of all one-one analytic mappings of $\mathfrak{E}_{p,q}$ $(p\neq q)$ on itself consists of the mappings

$$W=(A^{(p,p)}Z+B^{(p,q)})(C^{(q,p)}Z+D^{(q,q)})^{-1},$$

where $M=\begin{pmatrix}A^{(p,p)} & B^{(p,q)}\\ C^{(q,p)} & D^{(q,q)}\end{pmatrix}$ is an arbitrary element of $\Omega_{p,q}$. Previously the author had considered the case $p=q$.

J. Lehner (E. Lansing, Mich.).

See also: Linear Algebra: Parodi. **Theory of Algebraic Numbers:** Krasner. **Geometry of Numbers:** Turán. **Har-**

monic Functions, Convex Functions: Redheffer. **Special Functions:** Shenton (2 articles). **Integral Transforms:** Delange; Noble. **Ordinary Differential Equations:** Wittich. **Difference Equations, Functional Equations:** Montel. **TopologicalVector Spaces:** Sebastião e Silva. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Swerling; Shimoda, Watanabe and Terai; Shimoda. **Complex Manifolds:** Remmert; Stein; Grauert and Remmert.

Geometric Analysis

Hofsommer, D. J. A flux theorem. Appl. Sci. Res. B. 6 (1957), 446–448.

See also: Harmonic Functions, Convex Functions: Friedrichs; Morrey and Eells; Morrey. **Differential Geometry:** Karapetyan; Gallissot. **Manifolds, Connections:** Frankel; de Mira Fernandes.

Harmonic Functions, Convex Functions

Redheffer, R. M. On pairs of harmonic functions. Proc. Amer. Math. Soc. 8 (1957), 450–457.

Let u and v be a pair of functions of two real variables that are harmonic in a bounded region R. The author studies the extent to which properties of (u, v) depend on the functions being conjugate, that is to say on their satisfying the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$. For example, the classical maximum theorem stating that u^2+v^2 does not attain a maximum at an interior point is independent of conjugateness of u and v. Let α and β be the measures of conjugateness given by $\alpha=v_x^2+v_y^2-u_x^2-u_y^2$, $\beta=2(u_xv_x+u_yv_y)$; let M and m be constants with $u^2+v^2\geqq M$ on the boundary of R and $u^2+v^2=m$ at an interior point P of R; let D be the maximum diameter of R; and let d be the maximum distance of P from the boundary of R. It is proved that if $M\geqq m>0$ and u^2+v^2 has a minimum at P, then $\sup(\alpha^2+\beta^2)^{\frac{1}{2}}\geqq 2mD^{-2}\log(M/m)$, a result generalizing the minimum modulus theorem for conjugate harmonic functions. It is also proved that $\sup(u_x^2+u_y^2+v_x^2+v_y^2)\geqq 2(M-m)/d^2$, and that this result is best possible even for conjugate harmonic functions in the case of a circle R with centre P and with $m=0$. A final result gives an upper bound for $\inf(u_x^2+u_y^2+v_x^2+v_y^2)$.

F. F. Bonsall (Newcastle-upon-Tyne).

Yamasuge, Hirosi. Harmonic functions with two singular points. J. Inst. Polytech. Osaka City Univ. Ser. A. 8 (1957), 39–42.

Let $\mathfrak{M}$ be a closed orientable Riemannian manifold with a positive-definite metric $ds^2=g_{ik}dx^idx^k$, where the g_{ik} are holomorphic functions of $x^1, x^2, \cdots, x^n$. Theorem 1 asserts that, given any two points ξ_1 and ξ_2 in a small subdomain G, there exists a function φ such that $\Delta\varphi=0$ in $\mathfrak{M}-\xi_1-\xi_2$, and in a neighborhood of ξ_i $(i=1, 2)$,

$$\varphi(x)=\begin{cases}((n-2)\omega_n)^{-1}r^{2-n}(x,\xi_i)u_i+\log r(x,\xi_i)\cdot v_i+w_i, & n>2,\\ -(2\pi)^{-1}\log r(x,\xi_i)u_i+v_i, & n=2,\end{cases}$$

where ω_n is the surface area of the n-dimensional unit sphere, $r(x, \xi_i)$ is the geodesic distance between x and ξ_i, and u_i, v_i and w_i are holomorphic functions of x, with $u_i(\xi_i)=1$. A point is called stational if all first partial derivatives of φ vanish there; a stational point is non-degenerate if the determinant $|\partial^2\varphi/\partial x^i\partial x^k|$ is not zero there. The author considers relations between the number of stational points and the topological structure of $\mathfrak{M}$, for the case in which all stational points are non-degenerate. Here particular attention is given to the cases in which $n=2$ or $n=3$.

F. W. Perkins (Hanover, N.H.).

Choquet, Gustave; et Deny, Jacques. Modèles finis en théorie du potentiel. J. Analyse Math. 5 (1956–57), 77–135.

Les A. développent une théorie du potentiel sur un système X de m points x_i [théorie esquissée dans C. R. Acad. Sci. Paris 242 (1956), 222–225; MR 17, 1072]. Un noyau $G(x, y)$ est une fonction réelle $\geqq 0$ sur $X\times X$. Pour une mesure μ sur X chargeant x_i de la masse m_i, on définit le potentiel $G\mu(x)=\sum G(x, x_i)m_i$. G est dit dégénéré si le déterminant des $G(x_i, x_j)$ est nul. On distingue pour G les propriétés ou principes suivants. Principes de domination: D, exprimant pour des mesures $\geqq 0$ que $G\mu\leqq G\nu$ sur le support $S\mu$ de μ entraîne l'inégalité partout; D' (principe inverse) avec les inégalités opposées; D'' (principe faible), lorsque, pour μ de signe quelconque, $G\mu\geqq 0$ sur $S\mu$ entraine $G\mu\geqq 0$ partout. Principes de balayage: B, exprimant que pour toute $\mu\geqq 0$ et $E\subset X$ il existe $\mu'\geqq 0$ portée par E telle que $G\mu'=G\mu$ sur E, $G\mu'\leqq G\mu$ partout: B' (principe inverse), avec l'inégalité contraire; B'' (principe faible), avec la seule égalité sur E. Principe de l'enveloppe inférieure: I, exprimant que l'enveloppe inférieure de deux potentiels de mesures $\geqq 0$ est un potentiel de mesure $\geqq 0$; et principe analogue S de l'enveloppe supérieure.

Relations logiques entre ces principes: D et B sont équivalents seulement pour G régulier, c'est-à-dire, tel que, pour tout x où $G(x, x)=0$, $G(x, y)$ et $G(y, x)$ sont nuls quel que soit y; ils sont équivalents si G est non dégénéré; D et B entrainent I. Tout G non dégénéré satisfaisant à I est tel que $G(x, \tau y)$ satisfait à B pour une permutation unique τ de X. Si G n'est jamais nul, D' et B' sont équivalents; D' et B' entrainent I. Tout G non dégénéré et jamais nul satisfaisant à S est tel que $G(x, \tau y)$ satisfait à B' (et D') par une permutation unique. Si G non dégénéré jamais nul satisfait à B'', il satisfait à B ou B'.

Dualité. On appelle noyau adjoint G^* le noyau $G^*(x, y)=G(y, x)$. On dit que deux principes sont duaux si la validité de l'un pour un noyau équivaut à la validité de l'autre pour le noyau adjoint. Alors B, B', B'' sont duaux de D, D', D''.

Puis on donne des critères caractérisant les noyaux satisfaisant à B ou B', d'où la forme générale des noyaux non dégénérés satisfaisant à B: noyaux $A\sum_{p=0}^{\infty}S^p$ avec A tel que $A(x_i, x_j)=0$ si $i\neq j$, >0 si $i=j$, et S noyau $\geqq 0$ tel que la série converge. Il y a unicité de la représentation si les $S(x_i, x_i)$ sont nuls. Enfin on montre comment l'étude des noyaux dégénérés se ramène à celle des autres et on examine le cas des noyaux symétriques pour lesquels on introduit utilement les notions d'énergie mutuelle et de capacité et où B, par exemple, entraine que le noyau soit de type positif. Cette étude qui prépare celle des noyaux en espace localement compact comporte des démonstrations par récurrence sur le nombre de points, ce qui ne pourra s'étendre.

M. Brelot (Paris).

Choquet, Gustave. Sur les fondements de la théorie fine du potentiel. C. R. Acad. Sci. Paris 244 (1957), 1606–1609.

Poursuivant l'étude d'une note [mêmes C. R. 243 (1956), 635–638; MR 18, 295], l'auteur reprend l'étude et l'usage des potentiels continus, seulement, pour simplifier, en espace compact E. Rappelons qu'un noyau $G(x, y)\geqq 0$

(mesurable en y pour toute mesure) est dit régulier si, pour toute $\mu \geqq 0$, le potentiel $G\mu$ est fini continu quand sa restriction au support de μ l'est. On le dira très régulier s'il est fini continu hors de la diagonale Δ de $E \times E$, infini sur Δ, et si G et $\check{G}(x, y)$ $(=G(y, x))$ sont réguliers. D'abord, si G est régulier et fini continu hors Δ, il existe pour toute μ et tout ε un ouvert de $\check{G}$-capacité $<\varepsilon$ tel que la restriction de $G\mu$ à $C\omega$ soit finie continue. Les G très réguliers ont de nombreuses propriétés comme l'identité des ensembles de G et $\check{G}$-capacité extérieure nulle. Citons surtout (a) si $\mu_n \to \mu$ et si $G\mu_n$ tend vers $f(x)$ q.p. (c'est-à-dire, sauf sur un ensemble de capacité extérieure nulle), $G\mu = f$ q.p.; (b) diverses équivalences de la propriété que l'égalité q.p. de deux potentiels entraîne l'égalité des mesures, en particulier celle que les $\check{G}\mu$ continus forment un ensemble total dans l'espace $\mathscr{C}(E)$ des fonctions finies continues. *M. Brelot* (Paris).

Choquet, Gustave. Potentiels sur un ensemble de capacité nulle. Suites de potentiels. C. R. Acad. Sci. Paris **244** (1957), 1707–1710.

Deny avait caractérisé [mêmes C. R. **224** (1947), 524–525; MR **8**, 380] les ensembles E des infinis d'un potentiel classique comme des G_δ de capacité nulle. L'A. montre que la mesure $\mu > 0$ dont le potentiel est infini seulement sur E peut être choisie de façon à ne charger que E (ce qui n'était connu que pour un E fermé ou F_σ). Cela s'étend aux potentiels en $r^{-\alpha}(\alpha \geqq n-2)$. D'autre part, même en espace seulement localement compact métrisable, on peut pour des noyaux G généraux (en particulier dans le cas classique) trouver un potentiel $G\mu$ $(\mu \geqq 0)$ qui, sur un ensemble donné relativement compact de $\check{G}$-capacité extérieure nulle, soit égale à une fonction semi continue inférieurement quelconque donnée >0. Enfin on montre dans le cas classique que, de toute suite de mesures >0 $\mu_n \to \mu$, on ne peut pas toujours extraire une suite μ_{n_i} telle que son potential $G\mu_{n_i}$ tende vers $G\mu$ quasi-partout; un exemple donne μ_n chargeant un compact E, et chaque sous-suite ne converge pas vers $G\mu$ sur un ensemble de même capacité que E. *M. Brelot* (Paris).

Endl, Kurt. Sur des problèmes du type de Dirichlet utilisant les lignes de Green. C. R. Acad. Sci. Paris **244** (1957), 1705–1707.

Les lignes de Green (trajectoires orthogonales des surfaces de niveau de la fonction de Green G_{P_0}) dans un domaine euclidien ou un espace de Green permettent de poser un problème de Dirichlet comme suit: recherche d'une fonction harmonique admettant en un certain sens à la frontière une fonction-moyenne limite (radiale) sur l'ensemble $\mathscr{L}$ des lignes de Green. On sait qu'il y a unicité pour les fonctions dites "indifférentes". L'auteur montre que l'existence n'a lieu en outre que pour des données (radiales) qui soient sommables pour une certaine mesure abstraite sur $\mathscr{L}$. D'autre part on étudie le rapport des dérivées à la frontière du domaine $G_{P_0} > \lambda$ et selon une ligne de Green issue de P_0, de $G_{P_0}{}^{\lambda}$ et $G_P{}^{\lambda}$ (fonctions de Green du domaine $G_{P_0} > \lambda$). Il y a une certaine limite faible $(\lambda \to 0)$ qui sert à une représentation intégrale de la solution du problème précédent. *M. Brelot* (Paris).

Yamasuge, Hirosi. Maximum principle for harmonic functions in Riemannian manifolds. J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 35–38.

The author considers functions harmonic in a subdomain G of $\mathfrak{R}$, an n-dimensional analytic Riemannian manifold with positive-definite metric $ds^2 = g_{ik}dx^i dx^k$, where each g_{ik} is a holomorphic function of $x_1, x_2, \cdots, x_n$. The culminating theorem asserts that if φ is a non-constant function, harmonic in a bounded domain G and continuous in $\bar{G} = G + BG$ (where BG is the boundary of G), then φ attains its maximum and minimum at points of BG. As a corollary the author notes that if a function is harmonic in the whole of a compact manifold $\mathfrak{M}$ then it is constant. *F. W. Perkins* (Hanover, N.H.).

Matsushita, Shin-ichi. Laplacien local et la décomposition de F. Riesz. Proc. Japan Acad. **32** (1956), 436–440.

Retour sur les démonstrations de deux notes précédentes [C. R. Acad. Sci. Paris **241** (1955), 1252–1254, 1373–1375; MR **18**, 29]. *J. Deny* (Strasbourg).

Epstein, Bernard; and Scheerer, Anne. The existence of a generalized Green's function in the plane. J. Analyse Math. **4** (1955/56), 222–235.

Let Ω be a bounded domain of a two-dimensional euclidean space E^2. Then the expectation of the time $T(y)$ spent in Ω by a Brownian motion, starting from a point $y \in E^2$ at $t=0$, is divergent, viz.,

$$E(T(y)) = \int_0^\infty \int_\Omega (2\pi t)^{-1} \exp(-|r-y|^2/2t)\,dr\,dt = \infty,$$

in contrast to the higher dimensional case treated by M. Kac [Ann. Soc. Polon. Math. **25** (1952), 122–130; MR **15**, 31]. Thus the authors introduce a parameter $\varepsilon > 0$ as follows:

$$E(T_\varepsilon(y)) = (2\pi)^{-1} \int_\Omega \int_0^\infty t^{-1} \exp(-\varepsilon^2 t/2) \exp(-|r-y|^2/2t)\,dt\,dr$$
$$= (2\pi)^{-1} \int_\Omega K(\varepsilon|r-y|)\,dr.$$

They define $h^*(y, u, \varepsilon) = 1 + \sum_{n=1}^{\infty} (n!)^{-1}(-u)^n E^*(T_\varepsilon{}^n(y))$, where $((2\pi)^n/n!)E^*(T_\varepsilon{}^n(y))$ is the n-fold convolution of $K(\varepsilon|r-y|)$. It is shown that, for a suitable sequence of values of ε, the double limit $\lim_{\varepsilon \downarrow 0} \lim_{u \uparrow \infty} \log \varepsilon^{-1} h^*(y, u, \varepsilon)$ exists and defines a Green's function: it is logarithmically infinite at ∞, harmonic, and zero at regular boundary points of Ω. *K. Yosida* (Tokyo).

Schiffer, Menahem. Sur les rapports entre les solutions des problèmes intérieurs et celles des problèmes extérieurs. C. R. Acad. Sci. Paris **244** (1957), 2680–2683.

Relations are derived between pairs of functions which assume the same values or have the same normal derivaties on a closed, sufficiently smooth surface S and where one is harmonic in the exterior $\bar{D}$, the other in the interior D of S. In H, which is the linear space of the functions which are harmonic in D and which have continuous partial derivatives on $D+S$, three different metrics are introduced:

$$[f, g] = \int_S \int_S \frac{\partial f(z)}{\partial n_z} \frac{\partial g(\zeta)}{\partial n_\zeta} \frac{1}{r(z, \zeta)}\, d\sigma_z d\sigma_\zeta,$$

$$(f, g) = -\int_S \frac{\partial f(z)}{\partial n} g(z)\, d\sigma,$$

$$\{f, g\} = 4\pi(f, g) - [f, g].$$

If the four functions f, g, $\bar{f}$, $\bar{g}$, satisfy the conditions: $f \in H$, $g \in H$, $\bar{f} \in \bar{H}$, $\bar{g} \in \bar{H}$ ($\bar{H}$ is for $\bar{D}$ the space corresponding to H) $f = \bar{f}$ on S, $\partial g/\partial n = \partial \bar{g}/\partial n$ on S, then it is shown that $[f, g] = -\{\bar{f}, \bar{g}\}$; $(f, g) = -(\bar{f}, \bar{g})$; $\{f, g\} = -[\bar{f}, \bar{g}]$.

From these equalities one can derive two interesting inequalities:

$$4\pi(\bar{g}, \bar{g}) \geqq [g, g] + \frac{[f, g]^2}{[f, f]^2},$$

$$4\pi(\bar{f}, \bar{f}) \geqq \{f, g\} + \frac{\{f, g\}^2}{[g, g]^2}.$$

These inequalities are used to derive a variational principle for the functional $(\bar{g}, \bar{g})$ if $g(z)$ is given, and similarly for $(\bar{f}, \bar{f})$ if $f(z)$ is given. *U. W. Hochstrasser.*

Friedrichs, K. O. Differential forms on Riemannian manifolds. Comm. Pure Appl. Math. **8** (1955), 551–590.

In this paper the author gives a variational approach to the theory of harmonic forms of a Riemannian manifold close to the one presented in the two papers reviewed below. This method differs from that of Morrey and Eells, although related.

The space B of L_2 forms is introduced. This is the space of square integrable forms with the usual Hilbert space metric, which is denoted by (ω, ω), where ω is a differential form. The operators d and δ, the exterior derivative and co-derivative, are introduced on B as closed unbounded operators. Their domains are denoted by B_d and B_δ respectively. We shall ignore the degree of the form, and simply understand that B is the direct sum of the forms of degree zero up to degree n, the dimension of the manifold. It is first pointed out that $B_d \cap B_\delta$ admits a direct sum decomposition of the form $\overline{dB}_d + \overline{\delta B}_\delta + H$, where H is the space of forms both closed and co-closed (the harmonic forms). Each of the spaces $\overline{dB}_d$, $\overline{\delta B}_\delta$ and H is closed with respect to the norm $\|\omega\|_1 = \|\omega\| + \|d\omega\| + \|\delta\omega\|$.

The space B_1 is the space of strongly differentiable forms; that is, the forms whose components in every coordinate patch possess strong derivatives $\nabla_k \omega_{k_1 \cdots k_p}$. For any coordinate covering R, with patches R_σ, a norm is introduced by $\|\omega\| + \sum_\sigma \|\nabla\omega\|_{R_\sigma}$ and $\sum_\sigma \|\nabla\omega\|_{R_\sigma}$ is called the complete Dirichlet integral, $D_c(\omega)$. The space B_1 is complete in this norm. The main theorem of Section 4 is that $B_1 = B_d \cap B_\delta$. In particular, every form in H has strong derivatives. In Section 5, it is shown that there are positive constants C and K such that

$$D_c^2(\omega) \leqq C\|d\omega\|^2 + C\|\delta\omega\|^2, \quad \|\omega\|^2 \leqq K\|d\omega\|^2 + K\|\delta\omega\|^2.$$

The final remark in Section 5 is that the spaces dB_d and δB_δ are actually closed, which is the Kodaira decomposition theorem for closed manifolds. Actually it is proved in Section 6 that a continuous form ω in dB is the differential $d\eta$ of a continuous form in δB, and a continuous form ω in δB is the co-differential of a continuous from η in dB. Section 7 is devoted to deriving Hodge's theorem from the foregoing results. This shows that every cohomology class on a closed manifold is represented by a unique harmonic form. All of the results up to this point are shown under the weak assumption that the components of the metric tensor satisfy only a Lipschitz condition. The point in working under such general conditions is strongly made in Part III, where the author gives a simple derivation of the Hodge theorems for a manifold with boundary by the device of doubling a finite manifold with boundary to obtain a closed manifold with a metric which is symmetric under the involution. Doubling consists in taking two copies of the manifold with boundary and identifying corresponding points along the boundary, and the metric is given by symmetry. However, the metric may not possess continuous derivatives along the boundary. The Hodge theorems in this case assert that the absolute cohomology of a manifold with boundary is represented by harmonic forms with vanishing normal component, while the relative cohomology modulo the boundary is represented by harmonic forms with vanishing tangential component.

The emphasis of this paper is on methods. The final section is devoted to showing that there is a constant C such that

$$D_c(\omega)^2 \leqq C(\|d\omega\|^2 + \|\delta\omega\|^2)$$

for all forms in B_1 which lie in the orthogonal complement of H, and this is the original aim of the work. The theory of harmonic forms for real Riemannian manifolds is completely treated by these methods. As pointed out earlier this is related to the Morrey-Eells work, but is quite different from the integral equation methods of Duff, Spencer, Kodaira, Hodge and Bidal and de Rham. It is noted that the complete Dirichlet integral was used by Gaffney. *P. E. Conner* (Ann Arbor, Mich.).

Morrey, Charles B., Jr.; and Eells, James, Jr. A variational method in the theory of harmonic integrals. I. Ann. of Math. (2) **63** (1956), 91–128.

This paper treats the theory of harmonic integrals, in particular the derivation of Hodge's theorem, by a variational method. The treatment is similar to that of Friedrichs [see the paper reviewed above]. The orthogonal decomposition formula for the differential forms on a manifold, which was obtained by Kodaira for real analytic Riemannian manifolds [Ann. of Math. (2) **50** (1949), 587–665; MR **11**, 108], is proved by the authors under minimum differentiability hypotheses. Indeed, it is only required that the manifold be of class C_1^1.

The first chapter is devoted to the potential theory of functions in Euclidean space and to the study of the regularity of a weak solution of an elliptic system of differential equations. The second chapter actually concerns harmonic integrals on a manifold. The idea in earlier treatments of harmonic integrals was to make an extension of the integral equation methods used in the study of the potential theory of functions to differential forms. In this note the authors have extended the notion of functions of a class P_2 [J. W. Calkin, Duke Math. J. **6** (1940), 170–186; C. B. Morrey, ibid. **6** (1940), 187–215; MR **1**, 208, 209] to differential forms and have, by methods analogous to those used in the case of functions, derived Hodge's theorem.

A differential form is said to be of class L_2 (or P_2) on the manifold M^n if and only if its components in each coordinate system are in L_2 (or in P_2). As usual, the inner-product is introduced into L_2, making it a Hilbert space. This inner product is denoted (ω, η).

If $U = (U_1, \cdots, U_Q)$ is a finite open cover by coordinate patches $U_q = \theta_q(G_q)$, where each G_q is a Lipschitz domain in E^n, then the inner-product for two forms in P_2 is introduced as

$$((\omega, \eta))_U = (\omega, \eta) + \sum_{q=1}^{Q} \int_{G_q} \sum_{(i)} \sum_{\alpha=1}^{n} \omega^{(q)}_{(i)x^\alpha} \eta^{(q)}_{(i)x^\alpha} \, dx,$$

where $\omega^{(q)}_{(i)}$ and $\eta^{(q)}_{(i)}$ are the components of ω and η in U_q. This inner-product is shown to make P_2 into a Hilbert space. It is then shown that the exterior derivative and co-derivative d and δ are bounded operators from P_2 into L_2. The Dirichlet integral

$$D(\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega)$$

is a lower semi-continuous function with respect to weak convergence in P_2. If ω_i converges weakly in P_2 to ω_0, then ω_i converges strongly in L_2 to ω_0.

The main result needed for application of the method is as follows. For any coordinate covering U there are constants $K_U>0$ and L_U such that

$$D(\omega)\geqq K_U((\omega,\omega))_U-L_U(\omega,\omega).$$

The authors derive the decomposition formula after a proof of the above inequality. *P. E. Conner.*

Morrey, Charles B., Jr. A variational method in the theory of harmonic integrals. II. Amer. J. Math. **78** (1956), 137–170.

The methods developed in the paper reviewed above are extended to the study of harmonic forms on a compact Riemannian manifold with boundary. Sharp differentiability results for solutions of boundary value problems are obtained under very weak differentiability conditions on the manifold by using an armory of techniques similar to those applied in the author's study of general second-order elliptic and strongly elliptic systems [Contributions to the theory of partial differential equations, Princeton, 1957, pp. 101–159; MR **16**, 827]. Parallel theories of boundary value problems for odd and even forms are constructed on non-orientable manifolds. An analogue is established for the Kodaira decomposition theorem on manifolds with boundary, and new proofs are given under weak differentiability conditions of existence theorems due to Duff and Spencer [Ann. of Math. (2) **56** (1952), 128–156; MR **13**, 987] and Conner [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1151–1155; MR **16**, 357]. *F. Browder* (Paris).

Morrey, Charles B., Jr. A variational method in the theory of harmonic integrals. Comm. Pure Appl. Math. **9** (1956), 499–508.

This is a report on essentially the work contained in the two papers reviewed above. The results of these two papers are outlined here and a list of references is given.

The basis of the method is the extension of the notion of functions of class P_2 to differential forms. The corresponding space of r-forms, P_2, is given an inner product such that weak convergence in P_2 implies strong convergence in the usual space of L_2 forms. The inner-product for P_2 is defined for a given open covering of the manifold by Euclidean neighborhoods. It is pointed out that for any covering U there exists constants K_U and L_U such that if $D(\omega)$ denotes the Dirichlet integral,

$$D(\omega)\geqq K_U((\omega,\omega))_U-L_U(\omega,\omega),\ K_U>0,$$

for any form ω in P_2. The symbol $((\omega,\omega))$ refers to the P_2 inner-product, which is defined for the open cover U as

$$((\omega,\eta))=(\omega,\eta)+\sum_q\int_{G_q}\sum_{(i),\alpha}\omega_{(i)x^\alpha}\eta_{(i)x^\alpha}\,dx,$$

where $\omega_{(i)x^\alpha}=\partial\omega_{(i)}/\partial x^\alpha$ and G_q is the Euclidean domain of the coordinate patch U_q. This is the main lemma for the arguments. By using this method, the Kodaira decompositions of harmonic forms both on a closed manifold and on a finite manifold with boundary are obtained. The boundary value problems for harmonic forms on a manifold with boundary are all solved here. It should be pointed out that use of these methods allows one to do the analysis of real harmonic integrals under the minimum assumptions of differentiability. As mentioned earlier, this note is meant to be an outline, and for details the reader is given the necessary references. *P. E. Conner.*

Domar, Yngve. On the existence of a largest subharmonic minorant of a given function. Ark. Mat. **3** (1957), 429–440.

On sait que dans un domaine euclidien les fonctions sousharmoniques majorées par une fonction $F\geqq 0$ semi-continue supérieurement ont une enveloppe supérieure qui est sousharmonique si et seulement si cette enveloppe est localement bornée. L'A. cherche des conditions sur F pour qu'il en soit ainsi, et indique des extensions, en particulier en remplaçant "sousharmonique" par "sous-médiane", et "F semi-continue supérieurement" par "F mesurable". Il établit deux conditions suffisantes sur F; la plus simple à exprimer est que l'enveloppe est bornée si, dans R^k, $\int[\log^+ F]^{k-1+\varepsilon}d\omega$ ($d\omega$ mesure de Lebesgue) est finie pour un $\varepsilon>0$. On ne peut améliorer l'exposant. *M. Brelot* (Paris).

Hayashi, Kenji. On almost sub-biharmonic functions. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 133–140.

L'auteur dit que dans R^n une fonction localement sommable est presque sousbiharmonique si $\Delta\Delta u$ est une distribution de Schwartz $\leqq 0$. L'auteur introduit en chaque point x l'opérateur

$$\mathscr{L}_R=(1-\alpha^2)^{-1}(L_{\alpha R}-\alpha^2 L_R),\ 0<\alpha<1,$$

L_ρ moyenne sur la surface sphérique de rayon ρ, centre x. De même $\mathscr{A}_R$ avec la moyenne spatiale A_ρ. Pour que ce soit presque sousbiharmonique il faut et suffit que presque partout $\mathscr{L}_R u$ (ou $\mathscr{A}_R u$) majore u, ou $\mathscr{L}_R u\geqq\mathscr{A}_R u$, quel que soit R. On utilise des fonctions indéfiniment dérivables et la continuité du produit de composition. *M. Brelot.*

Giaccardi, Fernando. Su alcune disuguaglianze. Giorn. Mat. Finanz. (4) **1** (1955), 139–153.

The author gives a counterpart of Jensen's equality for convex functions

$$f\left(\frac{\sum_1^n p_k x_k}{\sum_1^n p_k}\right)<\frac{\sum_1^n p_k f(x_k)}{\sum_1^n p_k}<\frac{x_n f(x_1)-x_1 f(x_n)}{x_n-x_1}+\frac{(f(x_n)-f(x_1))\sum_1^n p_k x_k}{(x_n-x_1)\sum_1^n p_k}.$$

Several inequalities, more or less known, are obtained by specialization. *J. Aczél* (Debrecen).

Kurepa, Svetozar. Convex functions. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 89–94. (Serbo-Croatian summary)

It is a well-known result of Ostrowski [Jber. Deutsch. Math. Verein. **38** (1929), Abt. 1, 54–62] that if a convex function $f(x)$, defined on an open interval, is bounded from above on a set T of positive measure then $f(x)$ necessarily is continuous. The author now makes the interesting observation that the continuity of the convex function $f(x)$ follows from its boundedness on a set T such that the interior measure of $T+T$ is positive, where $T+T$ is the set of values $x+y$ with x and y in T. For example, T might be the Cantor set. It is pointed out, by example, that the boundedness of the convex function $f(x)$ on a set T such that the exterior measure of $T+T$ is positive does not imply the continuity of $f(x)$. *E. F. Beckenbach.*

See also: Partial Differential Equations: Cattabriga; Pucci; Lavrent'ev. **Fluid Mechanics, Acoustics:** Littman.

Special Functions

Aissen, M. I. Cyclically ordered sets. Canad. J. Math. 9 (1957), 406–412.

Ayant défini des ensembles $E_1, E_2, E_3, \cdots, E_k$ cycliquement ordonnés comme étant tels que si λ et μ sont des entiers $\leq k$, $E_\lambda \cap E_\mu = \emptyset$, $Z_J(E_\lambda) \leq Z_J(E_\mu)+1$ (le symbole $Z(A)$ désignant le nombre d'éléments de l'ensemble A et $Z_I(A)$ signifiant $Z(A \cap I)$, où I est un intervalle), l'auteur établit un théorème important relatif aux zéros d'un système de fonctions $\{f_n(x)\}$ satisfaisant à une relation de la forme $f_{n+k}(x) = a_n(x)f_{n+k-1}(x) - b_n(x)f_n(x)$. Il en déduit que pour une suite de polynomes orthogonaux $p_n(x)$ sur un intervalle $(-a, +a)$ l'ensemble des zéros positifs de p_n, p_{n+1}, p_{n+2} est cycliquement ordonné.

Des remarques particulières sont faites sur les zéros des polynomes d'Hermite et de Legendre. *R. Campbell.*

Toscano, Letterio. Relazioni e disuguaglianze su polinomi classici. Boll. Un. Mat. Ital. (3) **12** (1957), 71–79.

Utilisant la formule sommatoire de Darboux-Christoffel, l'auteur démontre plusieurs relations ou inégalités, qui ont des formes analogues pour 3 catégories de polynomes orthogonaux classiques: ceux d'Hermite, de Laguerre et de Legendre. Des résultats analogues sont établis pour les fonctions "de 2ème espèce" associées aux polynomes précédents. La fin de l'article est consacré particulièrement aux polynomes d'Hermite pour lesquels des nouvelles formules (sommations et inégalités) sont données ou mises en relation avec la fonction hypergéométrique. *R. Campbell* (Caen).

Carlitz, Leonard. On Laguerre and Jacobi polynomials. Boll. Un. Mat. Ital. (3) **12** (1957), 34–40.

Utilisant une formule sommatoire de Bateman fournissant un développement de polynomes de Jacobi d'abord, puis de Laguerre, du produit de 2 fonctions de Bessel

$$J_\alpha(z \sin\varphi \sin\theta) \times J_\beta(z \cos\varphi \cos\theta),$$

l'auteur démontre un riche ensemble de relations concernant ces polynomes et reprend parfois des formules connues en donnant aux paramètres des valeurs particulières. *R. Campbell* (Caen).

Lorch, Lee; and Szego, Peter. A singular integral whose kernel involves a Bessel function. Duke Math. J. **22** (1955), 407–418.

Lorch, Lee; and Szego, Peter. Corrections and a remark to: A singular integral whose kernel involves a Bessel functions. Duke Math. J. **24** (1957), 683.

This paper is concerned with a singular integral over the interval $(0, A)$, $A>1$, with kernel $\nu J_\nu(\nu t)$. It is shown that this kernel behaves in a sense like a Fejér kernel in $(0, 1)$ and like a Dirichlet kernel in $(1, A)$. Assuming that $t^\lambda f(t)$ is integrable in $(0, 1)$ for some λ, that $f(1-0)$ and $f(1+0)$ exist, and that $f(t)$ is of bounded variation in $[1, A]$ it is shown that

$$\lim_{\nu\to\infty} \nu \int_0^A f(t) J_\nu(\nu t)\,dt = \tfrac{1}{2}f(1-0) + \tfrac{1}{2}f(1+0).$$

Further, the authors prove that if A is replaced by $1+p(\nu)$, where $\nu[2p(\nu)]^{3/2} \to 3\sigma$ as $\nu\to\infty$, then the integral on the left converges to

$$\tfrac{1}{2}f(1-0) + \tfrac{1}{2}f(1+0) \int_0^\sigma [J_{\frac{1}{3}}(t) + J_{-\frac{1}{3}}(t)]\,dt.$$

This last result is then combined with a theorem of Cooke-Makai which asserts that the integrals of $t^{-\lambda}|J_\nu(t)|$ between successive zeros of $J_\nu(t)$ form a decreasing sequence for $\lambda \geq 0$, $\nu > -1$, or $\lambda \geq -\frac{1}{2}$, $\nu > \frac{1}{2}$, in order to calculate the limit of $\sup_x \nu^\lambda |\int_0^x t^{-\lambda} J_\nu(t)\,dt|$ as $\nu\to\infty$. This limit turns out to be equal to

$$\tfrac{1}{3} + \tfrac{1}{3}\int_0^c [J_{\frac{1}{3}}(t) + J_{-\frac{1}{3}}(t)]\,dt$$

where c is the least positive zero of $J_{\frac{1}{3}}(t) + J_{-\frac{1}{3}}(t)$. Finally the authors give upper and lower bounds for

$$\sup_x \left| \int_0^x t^{-\lambda} J_\nu(t)\,dt \right|$$

and show that the Lebesgue constants of the kernel $\nu J_\nu(\nu t)$ are unbounded. The last integral arose in recent work of A. Zygmund and the reviewer.

{Insert, as factor preceding the integral, ν^λ in (1) on p. 407 and ν in (10.1) on p. 414. On p. 410 the result incorrectly ascribed to Porter is actually due to C. Sturm [cf. pp. 173–175 in J. Math. Pures Appl. **1** (1836), 106–186].} *A. P. Calderón* (Cambridge, Mass.).

Saran, Shanti. On the generalised hypergeometric function of Appell's type. Bull. Calcutta Math. Soc. **45** (1953), 125–132.

Saran, Shanti. Hypergeometric functions of three variables. Ganita **5** (1954), 77–91 (1955).

The author defines ten functions of three variables analogous to the functions of two variables studied by Appell [Appell and Kampé de Fériet, Fonctions hypergéométriques et hypersphériques, Gauthier-Villars, Paris, 1926]. A typical one of these ten functions is

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \times \sum \frac{\Gamma(m+\alpha_1)\Gamma(n+p+\alpha_2)\Gamma(m+\beta_1)\Gamma(n+\beta_2)\Gamma(p+\beta_3)}{\Gamma(m+1)\Gamma(n+1)\Gamma(p+1)\Gamma(m+n+p+\gamma_1)} x^m y^n z^p$$

where m, n, and p in the triple series run from 0 to ∞. Regions of convergence and integral representations of these series are obtained as well as partial differential equations satisfied by them. A correction, inserting an omitted page, is printed in Ganita **7** (1956), 65.

{The author advises that this article is intended to replace the one listed above.} *R. G. Langebartel.*

Saran, Shanti. Relations between functions contiguous to certain hypergeometric functions of three variables. Ganita **5** (1954), 69–76 (1955).

L'auteur rappelle la définition précédemment donnée par lui [voir l'article analysé ci-dessus] des fonctions hypergéométriques de trois variables, qui sont au nombre de 10. Il étudie spécialement ici les relations existant entre une de ces fonctions et celles qu'on en déduit en changeant l'un des paramètres a en $a+1$ ou $a-1$, puis plus généralement en changeant a en $a+n$ (n entier). Il forme ensuite les équations aux dérivées partielles satisfaites par ces dix fonctions. *R. Campbell* (Caen).

Srivastava, Krishna Ji. On certain hypergeometric integrals. Ganita **7** (1956), 13–28.

Lauricella a défini 4 types de fonctions hypergéométriques de 3 variables; ensemble complété par 10 autres fonctions, introduites par Shanti Saran [voir la deuxième analyse ci-dessus].

L'auteur donne ici les représentations intégrales de ces 14 types de fonctions. *R. Campbell* (Caen).

Singh, V. N. Certain generalized hypergeometric identities of the Rogers-Ramanujan type. Pacific J. Math. **7** (1957), 1011–1014.

Se référant à un récent travail de Alder [même J. **4** (1954), 161–168; MR **15**, 856] l'auteur retrouve les mêmes résultats comme cas limites d'une transformation appliquée à la série hypergéométrique par Sears [Proc. London Math. Soc. (2) **52** (1951), 467–483; MR **13**, 33]. La méthode donne une forme plus simple aux polynomes introduits précisément par Alder. *R. Campbell* (Caen).

Shenton, L. R. A determinantal expansion for a class of definite integral. IV. Proc. Edinburgh Math. Soc. (2) **10** (1957), 153–167.

Continuing earlier work [same Proc. **9** (1953), 44–52; **10** (1954), 78–91, 134–140; MR **15**, 781; **16**, 575; **17**, 844], the author relates the generalized convergents of

$$F(z_1, \cdots, z_n)=\int_0^\infty \frac{d\psi(x)}{f_n(x)},$$

where $f_n(x)=(x+z_1)\cdots(x+z_n)$, to the convergents of the continued fraction expansion of $\int_0^\infty d\psi(x)/(x+z)$. Under the assumption that the Stieltjes moment problem for $\psi(x)$ has a unique solution, he also obtains four kinds of generalized continued fraction expansions of $F(z_1, \cdots, z_n)$.

A. Erdélyi (Pasadena, Calif.).

Shenton, L. R. A determinantal expansion for a class of definite integral. V. Recurrence relations. Proc. Edinburgh Math. Soc. (2) **10** (1957), 167–188.

This is the concluding part [for parts I–IV and notation see the preceeding review]. The author gives further discussion of his generalized continued fraction expansion of $F(z_1, \cdots, z_n)$, especially in the cases $n=2$ and 3. Assuming that the moment problem corresponding to $\psi(x)$ is determined, he gives a fifth order recurrence relation for the generalized convergents for $n=2$. *A. Erdélyi.*

See also: Analytic Theory of Numbers: Fine; Mitrović. **Theory of Algebraic Numbers:** Gundlach. **Functions of Complex Variables:** Specht; Denjoy (3 articles). **Sequences, Series, Summability:** Rudin. **Approximations, Orthogonal Functions:** van Rossum (2 articles). **Partial Differential Equations:** Babakova; Kozyrev. **Numerical Methods:** Cerrillo and Kautz; Milne-Thomson.

Sequences, Series, Summability

Rudin, Walter. Laplace series and sets of logarithmic capacity zero. Proc. Amer. Math. Soc. **6** (1955), 915–918.

The author shows that, given a subset E of the unit sphere of Euclidean 3-space of positive logarithmic capacity, there exists a Laplace series $\sum Y_n(P)$ which is $(C, 1)$ summable to zero outside E and such that $\sum Y_n(P)/n(n+1)$ is the Laplace series of a continuous function. Further, E is of positive capacity if and only if it carries a non-vanishing measure whose Laplace-Stieltjes series $\sum Y_n(P)$ is such that $\sum Y_n(P)/n(n+1)$ is the Laplace series of a continuous function. These facts are then used to show that sets of zero capacity cannot be replaced by sets of positive capacity in some results established earlier by the author [Trans. Amer. Math. Soc. **68** (1950), 278–286, 287–303; MR **11**, 663, 430].

A. P. Calderón (Cambridge, Mass.).

Stipanić, Ernest. Due teoremi sulle serie a termini positivi. Boll. Un. Mat. Ital. (3) **12** (1957), 50–56.

Let $\sum c_n$ and $\sum d_n$ be respectively convergent and divergent series of positive terms; let $r_n=\sum_{n+1}^\infty c_k$, $D_n=\sum_1^n d_k$; similarly for primed letters. The author shows that if $c_n/c_n' \to g>0$ and $c_n'/r_{n-1}' \leqq c_n/r_{n-1}$ then

$$\sum \{(c_n/r_{n-1})-(c_n'/r_{n-1}')\}$$

converges, while if $c_n/c_n' \to 0$ then $\sum |(c_n/r_n)-(c_n'/r_n')|$ diverges. Similarly, if $d_n/d_n' \to l>0$ and $d_n'/D_n' \leqq d_n/D_n$ then $\sum \{(d_n/D_n)-(d_n'/D_n')\}$ converges, while if $d_n/d_n' \to 0$ then $\sum |(d_n/D_{n-1})-(d_n'/D_{n-1}')|$ diverges.

R. P. Boas, Jr. (Evanston, Ill.).

Copping, J. Conditions for a K-matrix to evaluate some bounded divergent sequences. J. London Math. Soc. **32** (1957), 217–227.

For K-matrices (convergence-preserving Toeplitz matrices), the author formulates a theorem which states that, to a certain extent, the early and the late terms can be dropped from the series $\sum_k a_{nk}x_k$ which determines the nth element of the transform Ax; naturally, the extent depends on the matrix and on the least monotonic majorant of the sequence $\{|x_k|\}$. The theorem is used to prove, among other propositions, the following partial converse of a theorem of Darevsky: If the K-matrix A evaluates some divergent sequence whose growth is sufficiently slow (in a sense formulated in terms of properties of A), then there exists a bounded divergent sequence which is evaluated by A. *G. Piranian.*

Reĭmers, Ė. Mean value theorem for absolute summation. Uč. Zap. Tartu. Gos. Univ. **42** (1956), 113–134. (Russian. Estonian summary)

A Toeplitz matrix $A=(a_{nk})$ is said to satisfy a mean-value theorem (MVT) in a sequence space R if, for each sequence s in R, each partial sum $\sum_{k=0}^{k} a_{nk}s_k$ has a modulus less than a constant multiple $K|\sum_{k=0}^{m} a_{mk}s_k|$ of a conveniently chosen "complete sum" with $m \leqq n$ (here K is to depend only on the matrix). The concept of the MVT has been developed mostly by Jurkat and Peyerimhoff [see, e.g., Math. Z. **55** (1951), 92–108; MR **13**, 934]. The present paper formulates the analogous definition of the MVT for absolute summability, and develops various results, too complicated to be stated here. These results concern Abschnittskonvergenz, inclusion of ordinary and of absolute convergence fields, absolute summability of Cauchy products, and Riesz means. *G. Piranian.*

★ **Lyttkens, Sonja. The remainder in Tauberian theorems.** Dissertation, University of Uppsala, 1956. 7 pp.

A summary of the two papers reviewed in MR **15**, 858 and MR **18**, 895.

See also: Combinatorial Analysis: Gould. **Special Functions:** Singh. **Approximations, Orthogonal Functions:** Tandori.

Approximations, Orthogonal Functions

Berman, D. L. On a new method of constructing Weierstrass's interpolation formulae. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 679–682. (Russian)

In this note a new solution is given for two problems in the theory of approximation which were stated and solved by S. N. Bernstein [Collected works, vol. 1, Izdat. Akad.

Nauk SSSR, Moscow, 1952, p. 520; vol. 2, ibid., p. 130; MR 14, 2; 16, 433]. The new Weierstrass's interpolation formula given by the author has the form:

$$U_{n,m}(x)=n^{-1}\sum_{j=1}^{n} f(x_j^{(n)})[T_{n,m}(\theta+\theta_j)+T_{n,m}(\theta-\theta_j)],$$

where $f(x)$ has to be a continuous function defined on $[-1, 1]$ and

$$T_{n,m}(t)=\tfrac{1}{2}+\sum_{k=1}^{2n-m-1}\cos kt+\sum_{k=2n-m}^{m}\lambda_k^{(m)}\cos kt;$$

the $\lambda_k^{(m)}$ are any numbers satisfying $\lambda_k^{(m)}+\lambda_{2n-k}^{(m)}=1$, $k=2n-m, \cdots, (n-1)$, and $x=\cos\theta$, $\theta_j=(2j-1)\pi/2n$. In a number of theorems stated without proof the properties of this approximation formula are given. They are: (1) $U_{n,m}(x_j^{(n)})=f(x_j^{(n)})$, where $x_j^{(n)}=\cos\theta_j$; (2) if $f(x)$ is any polynomial of degree $<2n-m$ then $U_{n,m}(x)\equiv f(x)$; (3) if for any m

$$|\lambda_k^{(m)}|<c_1/\ln k,\ |\Delta\lambda_{k-1}^{(m)}|<c_2,$$

$$\sum_{k=2n-m}^{m-1}|\Delta^2\lambda_k^{(m)}|<c_3/(2n+1)$$

($\Delta\lambda_k$ and $\Delta^2\lambda_k$ being the first and second finite differences), where c_1, c_2, c_3 are constants not depending on k and n, and $1+\delta_1<m/n<1+\delta_2$, where δ_1 and δ_2 are arbitrary positive fixed numbers, then $\max_{-1\le x\le 1}\|U_{n,m}(x)-f(x)\|\to 0$ with $m\to\infty$, $n\to\infty$; (4) if $f(x)$ is a polynomial of degree $<2n-m$ satisfying $|f(x_j^{(n)})|\le 1$ $(j=1, 2, \cdots, n)$, then, in any point $x\in[-1, 1]$,

$$|f(x)|\le\frac{1}{n}\inf\sum_{j=1}^{n}|T_{n,m}(\theta+\theta_j)+T_{n,m}(\theta-\theta_j)|.$$

Two other theorems which are variations of (3) with different assumptions on the sequence $\lambda_k^{(m)}$ are also stated. {On p. 680 there is a misprint on line 12; it should read $\phi(m)\to\infty$ for $m\to\infty$}. *U. W. Hochstrasser.*

Paszkowski, S. On the Weierstrass approximation theorem. Colloq. Math. 4 (1957), 206–210.

Let C be the Banach space of continuous functions on a closed interval I. A sequence $\{\mu_n\}$ is called a Markoff sequence if a "polynomial" $\sum_k^n a_k\mu_k$ has at most n different roots. W_n denotes the set of polynomials of degree n. Suppose that ξ in C and the set T consisting of m points $t_1, \cdots, t_m$ on I be given. $\varepsilon_n(\xi)=\inf\|\xi-\psi\|$ with ψ ranging over W_n; and $\varepsilon_n(\xi; T)=\inf\|\xi-\psi\|$ with ψ ranging over $W_n(\xi; T)$, where $W_n(\xi; T)$ denotes the set of polynomials ϕ such that $\phi(t_i)=\xi(t_i)$, $1\le i\le m$. The first theorem proved here, with a neat proof, is that there exists a constant s depending only upon T such that $s\varepsilon_n(\xi)\ge\varepsilon_n(\xi; T)\ge\varepsilon_n(\xi)$ holds. The second theorem deals with ordinary polynomials. Suppose that $T=T_n$ itself changes and the number of points $m=m_n$ increases subject to a condition $\min_{i\ne j}|t_i-t_j|\ge\text{constant}/m_n$ as n goes to infinity with an order of

$$m_n=o\left(\frac{|\log\varepsilon_n(\xi)|}{\log|\log\varepsilon_n(\xi)|}\right);$$

then $\lim\varepsilon_n(\xi, T_n)=0$. The following questions are proposed. What is $\liminf\varepsilon_n(\xi; T)/\varepsilon_n(\xi)$; or does $\lim\varepsilon_n(\xi; T)/\varepsilon_n(\xi)=1$? *H. Yamabe* (Minneapolis, Minn.).

Paszkowski, S. On the number of affinically different sets. Colloq. Math. 4 (1957), 211–215.

Given a set of points A_s on a straight line, with the coordinates 0, 1, $\cdots$, s, let $A_{s,r}$ denote the family of all subsets of A_s consisting of $r+1$ points, $r\ge 1$. Let $A_{s,r}$ be divided into classes on the basis that two sets belong to the same class if and only if one of them can be affinely transformed into the other. A formula is given for the number of such classes. *K. Chandrasekharan.*

Arató, M.; and Rényi, A. Probabilistic proof of a theorem on the approximation of continuous functions by means of generalized Bernstein polynomials. Acta Math. Acad. Sci. Hungar. 8 (1957), 91–98.

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables. Let X_k be non-negative and have finite positive mean $M(X_k)$ and variance $D^2(X_k)$. We suppose that (i) $\lim_{k\to\infty}M(X_k)=0$; (ii) $\sum_{k=1}^{\infty}M(X_k)=\infty$; and (iii) $\lim_{k\to\infty}D^2(X_k)/M(X_k)=0$. We set $p_{n,0}(x)=P(X_n+\cdots+X_1<x)$, $p_{n,n}(x)=P(X_n\ge x)$, and $p_{n,k}(x)=P(X_n+\cdots+X_{k+1}<x\le X_n+\cdots+X_k)$. It is shown that if $f(x)$ is continuous for $0\le x\le\infty$, that is, $f(+\infty)$ exists, and if

$$B_n[f(x)]=\sum_{k=0}^{n}f\{\sum_{j=k+1}^{n}M(X_j)\}p_{n,k}(x),$$

then $\lim_{n\to\infty}B_n[f(x)]=f(x)$ uniformly for $0\le x\le\infty$. As a special case assume that the density function of X_k is 0 for $x<0$ and $a_k e^{-a_k x}$ for $x\ge 0$, where the a_k's are positive constants satisfying: $\lim_{k\to\infty}a_k=\infty$ and $\sum_{k=1}^{\infty}1/a_k=\infty$. The assumptions (i), (ii), and (iii) given above are satisfied and one obtains in this way a new proof of the approximation theorem for generalized Bernstein polynomials, first proved by the reviewer and D. V. Widder [Duke Math. J. 16 (1949), 433–438; MR 11, 29] and A. O. Gelfond [Izv. Akad. Nauk SSSR Ser. Mat. 14 (1950), 413–420; MR 12, 332]. *I. I. Hirschman* (St. Louis, Mo.).

Price, J. J. Certain groups of orthonormal step functions. Canad. J. Math. 9 (1957), 413–425.

L'étude est relative à un système orthonormé de fonctions généralisant celui des fonctions de Walsh, dans un sens indiqué déjà par Fine [Trans. Amer. Math. Soc. 65 (1949), 372–414; MR 11, 352], puis par Chrestenson [Pacific J. Math. 5 (1955), 17–31; MR 16, 920]. Ces auteurs ont fait remarquer que le système de Walsh était rattachable à un certain groupe abélien compact (produit de groupes d'ordre 2) et que le système obtenu en remplaçant les groupes d'ordre 2 de la définition de Walsh par des groupes d'ordre α fini jouissait de propriétés tout à fait analogues.

L'objet du présent travail est de montrer que tout devient très différent quand l'ordre α du groupe considéré cesse d'être borné. Les fonctions envisagées sont elles-mêmes des cas particuliers d'un système introduit par Ohkuma [Tôhoku Math. J. (2) 5 (1953), 166–177; MR 15, 867] et sont définies ainsi: Soit $\{n_1, n_2, \cdots, n_k\}$ une suite d'entiers $(n_i\ge 2)$; et si $p_0=1$, $p_k=n_1n_2n_3\cdots n_k$, soit $I(r, k)$ l'intervalle $r/p_k\le x<(r+1)/p_k$ et $\chi_r^{(k)}(x)$ la fonction caractéristique de $\mathbf{U}I(j, k)$, où j (entier) $\equiv r \pmod{p_k}$; on pose $\phi_{k-1}(x)=\sum_0^{p_k-1}\omega_k{}^r\chi_r^{(k)}(x)$ $(\omega_k=e^{2\pi i/n_k})$. Les théorèmes relatifs aux développements ordinaires de Fourier d'une fonction continue qui restent vrais pour les systèmes de Walsh et de Chrestenson cessent, en partie, de l'être pour ce système. Ainsi, ceux sur l'ordre de grandeur des coefficients du développement de $f(x)$, si $f(x)\in\text{Lip}\,\alpha$ on si $f(x)\in\text{BV}$; ceux sur la sommabilité $|(C, 1)$. Des précisions très curieuses et très intéressantes sont fournies sur ce dernier point. *R. Campbell* (Caen).

Watari, Chinami. A generalization of Haar functions. Tôhoku Math. J. (2) **8** (1956), 286–290.

This paper contains the construction of generalized Haar functions H_α which is analogous to the construction of the generalized Walsh functions by Chrestenson [Pacific J. Math. **5** (1955), 17–31; MR **16**, 920]. Among other results, an analog of Theorem 1 of Chrestenson's paper is stated and proved asserting the convergence of the H_α-expansion of a function to the value of that function at a point of continuity (uniformly in every closed interval of continuity). *G. K. Kalisch* (Minneapolis, Minn.).

★ **van Rossum, Hermanus. A theory of orthogonal polynomials based on the Padé table.** Thesis, University of Utrecht, 1953. 76 pp.

van Rossum, H. Systems of orthogonal and quasi orthogonal polynomials connected with the Padé table. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indag. Math. **17** (1955), 517–525, 526–534, 675–682.

The Padé table for a given pair of power series $S_1(x)$ and $S_2(x)$ and the systems of orthogonal and quasi-orthogonal polynomials connected with it are studied. Also contained here is a summary of results previously obtained by the author [thesis, listed above]. All numbers considered are real. I. It is shown for a regular Padé pair $Q_{\mu\nu}^{(1)}(x)$, $Q_{\mu\nu}^{(2)}(x)$, where $Q_{\mu\nu}^{(1)}$ has degree $\leqq \mu$, $Q_{\mu\nu}^{(2)}$ degree $\leqq\nu$, all common factors removed, and for

$$S_1(x)/S_2(x)=a_0+a_1x+a_2x^2+\cdots,$$

that the polynomials $x^nQ_{n,k+n}^{(1)}(x^{-1})$ $(n=0, 1, 2, \cdots)$ are orthogonal with respect to the sequence $a_{k+1}, a_{k+2}, \cdots$. An analogous theorem, that the polynomials $x^{k+n}Q_{n,k+n}^{(2)}(x^{-1})$ $(n=0, 1, 2, \cdots)$ are quasi-orthogonal of order k, also holds. II. A number of theorems concerning orthogonal and quasi-orthogonal systems of polynomials are obtained. In particular, if $S_1(x)=c_0+c_1x+c_2x^2+\cdots$, $c_0\neq 0$, $S_2(x)\equiv 1$, special cases of the theorems in (I) are obtained which were previously proved by means of continued fractions by the author (op. cit.). III. The theory developed above is applied to a study of the Padé table and orthogonality relations of Laguerre, Hermite, Jacobi, ultraspherical, and generalized Bessel polynomials. *E. Frank* (Chicago, Ill.).

Tandori, Károly. Sur les constantes de Lebesgue des systèmes de fonctions orthogonales et normées. C. R. Acad. Sci. Paris **244** (1957), 1128–1130.

If $\varphi_n(x)$ is a system of functions orthonormal over the interval (a, b) and

$$L_n^\alpha(t)=\int_a^b\left|\frac{1}{A_n{}^\alpha}\sum_0^n A_{n-\nu}^\alpha\varphi_\nu(x)\varphi_\nu(t)\right|dx,\quad A_n{}^\alpha=\binom{n+\alpha}{n},$$

are the Lebesgue constants of the system associated with the method of summability (C, α), and $\nu(n)$ is such that $\sum_2^\infty[\nu(n)^2n\log n]^{-1}<\infty$, then $L_n{}^\alpha(t)=o([\nu(n)^2n\log n]^{\frac{1}{2}})$ almost everywhere in t for each $\alpha\geqq 0$. If the φ_n are uniformly bounded then $L_n{}^\alpha(t)=O(n^{\frac{1}{2}})$ everywhere in $[a, b]$ for each $\alpha\geqq 0$. These results are best possible. *A. P. Calderón.*

See also: Functions of Complex Variables: Levin. **Trigonometric Series and Integrals:** Koosis.

Trigonometric Series and Integrals

Yano, Kenji. On the jump functions. Kôdai Math. Sem. Rep. **9** (1957), 1–11.

The author extends and strengthens some theorems of Szász, H. C. Chow, Maruyama Mohanty and Nanda. Here are some of the results. Let $\bar\sigma_n{}^\alpha(x)$ denote the (C, α) means of the conjugate of the Fourier series of $f(x)$, and $\varphi(t)=f(x+t)-f(x-t)-l$. If $\int_0^t\varphi(u)du=o(t)$ and $\int_0^t|\varphi(u)|du=O(t)$ as $t\to 0$, and $\alpha>0$ and $p>0$, then $\bar\sigma_{[pn]}{}^\alpha(x)-\bar\sigma_n\|(x)=l/\pi\log p+o(1)$ as $n\to\infty$. Other results concerning the behaviour of the sequence $\bar\sigma_m{}^\alpha(x)-\bar\sigma_n{}^\beta(x)$ on various assumptions on $f(x)$, α, β and the sequence $m=m(n)$ are established. The following theorem is also proved. Under the above assumptions the sequence $n(b_n\sin nx-a_n\cos nx)$ is summable $(C, \alpha+1)$ to l/π, where a_n and b_n are the Fourier coefficients of $f(x)$. *A. P. Calderón.*

Verblunsky, S. An outline of the theory of F_2 series. Amer. J. Math. **77** (1955), 628–654.

The author continues previous work on F.A. series [Rend. Circ. Mat. Palermo (2) **3** (1954), 89–105; MR **15**, 953]. Let $A(z)=z^2+a$ and $B(z)=bz$ where a and b are real, and let the roots μ_k of $A(z)\sin\pi z-B(z)\cos\pi z=0$ be all real and simple. Let λ_k be the residue of

$$\tfrac{1}{2}(A\cos\pi z-B\sin\pi z)/(A\sin\pi z-B\cos\pi z)$$

at μ_k. Then associated with $f\in L(\alpha, \alpha+2\pi)$ there is the F_2A series

$$(*)\qquad \tfrac{1}{2}a_0+\sum_1^\infty(a_k\cos\mu_kx+b_k\sin\mu_kx),$$

where

$$a_k-ib_k=2\lambda_k\int_\alpha^{\alpha+2\pi}f(x)e^{-i\mu_kx}dx.$$

If A and B are interchanged one obtains an F_2B series. The theory of such series is developed in the case when $b^2+4a>0$. The main results for the F_2A series are the following. If (*) is an F_2A over an interval $(\alpha, \alpha+2\pi)$, then except for the two first terms it is an F_2A over any interval $(\beta, \beta+2\pi)$. For given α the a_k, b_k $(k>1)$ and a suitable linear homogeneous condition determine a_0, a_1 and b_1 uniquely. An F_2A series is almost everywhere $(C, 1)$ summable to a function of which it is the Fourier series in the sense of almost periodic functions. A theorem of the Young-Hausdorff type holds for F_2A series. Finally, it is shown that if $\sum_1^\infty(a_k\cos\mu_kx+b_k\sin\mu_kx)$ converges to zero in $(-\pi, \pi)$ then it is the differentiated series of the F_2A series of a constant over $(-\pi, \pi)$. *A. P. Calderón.*

Baiada, Emilio. La limitatezza, gli zeri di una funzione e i coefficienti di Eulero-Fourier. Rend. Circ. Mat. Palermo (2) **4** (1955), 91–111.

The author uses the results of Toeplitz and Carathéodory concerning the coefficients of the Fourier-Stieltjes series of a non-decreasing function and some theorems of Ghizzetti concerning the coefficients of a function between given bounds in order to characterize functions which are i) non-negative, ii) with values between 0 and 1, iii) positive and bounded away from zero. Further the author gives necessary and sufficient conditions (also in terms of the Fourier coefficients) in order that, given a function of bounded variation, it be possible to decompose $(0, 2\pi)$ into m intervals over each of which the function is non-negative or non-positive. It is shown that a theorem of Hurwitz concerning the number of zeros of a continuous function of bounded variation whose first $2k+1$ Fourier coefficients vanish is a consequence of these results. *A. P. Calderón* (Cambridge, Mass.).

Krabbe, G. L. Spectra of convolution operators on L^p and rings of factor-sequences. Quart. J. Math. Oxford Ser. (2) **8** (1957), 1–12.

This paper considers the space of bounded sequences $c=\{c_n\}$, $-\infty<n<\infty$, such that

$$\text{l.u.b.}_N \sum_N |c_{n+1}-c_n| = M < \infty,$$

where $\sum_N$ is extended over $2^N \leq |n| \leq 2^{N+1}$, $N=0, 1, 2, \cdots$, and introduces the norm $\|c\| = \|c_n\|_\infty + M$. According to a theorem of Marcinkiewicz [Studia Math. **8** (1939), 78–91], if a_n are the Fourier coefficients of a function f in L^p, $1<p<\infty$, then the $a_n c_n$ are the Fourier coefficients of a function Tf in L^p. The author proves that $\varpi(c)=\sigma(T)=\sigma(c)$, where $\varpi(c)$ is the closure of the set $\{c_n\}$, $\sigma(T)$ and $\sigma(c)$ are the spectra of T and c respectively. He further shows that $\|c\|_\infty \leq \|T\|_p \leq k_p \|c\|$. *A. P. Calderón.*

Pistoia, Angelo. Sulla serie $\sum_1^\infty k\ \alpha_k |\sin(\lambda_k x)|^{\nu_k}$. Boll. Un. Mat. Ital. (3) **12** (1957), 41–45.

In the series of the title α_k, λ_k are nonnegative, $\nu_k \geq q > 0$, and $0 \leq x < \infty$. A necessary and sufficient condition that the series converges for almost every x to a function $\phi(x)$ with $x^{-p}\phi(x) \in L(0, \infty)$ is that $\sum \alpha_k \lambda_k^{p-1} \nu_k^{-1}$ converges. *R. P. Boas, Jr.* (Evanston, Ill.).

Domar, Y. On spectral analysis in the narrow topology. Math. Scand. **4** (1956), 328–332.

Beurling's narrow topology [Acta Math. **77** (1945), 127–136; MR **7**, 61] for the space of bounded continuous functions on the real line is based on neighborhoods

$$\{\psi: \max_{|x|<a} |\psi(x)-\phi(x)| + \|\psi-\phi\| < \varepsilon\}$$

of the fixed function ϕ ($\varepsilon>0$, $a>0$). Beurling has proved (loc. cit.) by using the theory of analytic functions that, if ϕ is bounded, uniformly continuous, and not identically zero, then the narrow closure of the linear space spanned by translates of ϕ contains at least one function of the form $e^{i\lambda x}$. The author gives a short and elementary proof of this theorem. He notes also that his proof can be at once generalized to the case of an arbitrary locally compact abelian group. *E. Hewitt* (Seattle, Wash.).

Koosis, Paul. Approximation of certain functions by exponentials on a half line. Proc. Amer. Math. Soc. **8** (1957), 428–435.

For functions $f(x)$ continuous and integrable on $(-\infty, +\infty)$ which vanish on $(-\infty, 0]$, the author considers the topology of uniform approximation on compact subsets of $[0, +\infty)$. Such $f(x)$ is mean periodic if at least one continuous function vanishing outside an interval $[0, L]$ is not approximated by the linear combinations of the translates $f_h(x)=f(x+h)$, $h>0$. Let L_f be the infimum of these L, and let X_f denote the set of functions of the form $x^n e^{-iax}$ ($n=0, 1, \cdots$; $\text{Im}\, a \leq 0$) which are approximable by linear combinations of the f_h. The following theorem of "harmonic synthesis" is proved. If f is mean periodic then, for $h>L_f$, f_h is approximable by linear combinations of the functions of X_f. This is not exact if $h<L_f$. The proof uses properties of the Fourier transform of f in the complex plane and a variation of a "division theorem". Let $F(z)=G(z)/\int_0^A e^{-izt}dm(t)$, where $F(z)$ is bounded for $\text{Im}\, z \geq 0$, and $G(z)$ is bounded for $\text{Im}\, z \leq k$ and each $k>0$. Then for each $\varepsilon>0$,

$$|F(z)| \leq K(\varepsilon) \exp((A+\varepsilon)|z|).$$

A similar approximation theorem, but with different topology, was given by B. Nyman [Thesis, Univ. of Uppsala, 1950; MR **12**, 108]. *G. G. Lorentz.*

See also: Functions of Complex Variables: Levin. **Approximations, Orthogonal Functions:** Price; Tandori; Watari. **Difference Equations, Functional Equations:** Montel.

Integral Transforms

Gaffey, William R. A real inversion formula for a class of bilateral Laplace transforms. Pacific J. Math. **7** (1957), 879–883.

The following inversion formula is established. If $\phi \in L(-\infty, \infty)$ and f is its bilateral Laplace transform; if ϕ is continuous at c; if $\int_{-\infty}^{x} \phi(z)dz \leq A \exp(-dx^{2+\alpha})$, with A, d, α positive; and if ε is between $\max(1/3, 1/(2+\alpha))$ and $\frac{1}{2}$; then

$$\phi(c) = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{c+k^\varepsilon}\right)^{k+1} \frac{d^k}{dt^k}[f(t)\exp(-tk^\varepsilon)],$$

where the derivative is evaluated at $t=k/(c+k^\varepsilon)$. The formula was suggested by a formal limiting process starting from the Post-Widder inversion formula. *R. P. Boas, Jr.* (Evanston, Ill.).

Delange, Hubert. Sur les singularités des fonctions définies par des intégrales de Laplace. Rend. Sem. Mat. Fis. Milano **26** (1954–55), 88–102 (1957).

In this lecture the author states several theorems about singular points of Laplace-Stieltjes integrals $f(s)=\int_0^\infty e^{-st}d\alpha(t)$, generalizing known results for Dirichlet series. New methods of proof are required for some of them. Some simple special cases of the deeper results are as follows. If $\alpha(t)=\int_0^t e^{t\psi(u)}dv(u)+\text{const.}$, $v(t)$ real and nondecreasing, $\psi(t)$ real, continuous, and such that $\psi(t+u)-\psi(t)\to 0$ as $t\to\infty$, then the real point σ_c on the line of convergence is a singular point of f. If the last hypothesis on ψ is replaced by $|\psi(t'')-\psi(t')| \leq k|t''-t'|$, then f has at least one singular point on $[\sigma_c - ik, \sigma_c + ik]$. *R. P. Boas, Jr.*

Noble, M. E. Boundary behaviour of a class of Laplace-Stieltjes transforms. J. London Math. Soc. **32** (1957), 156–170.

The author's earlier study of overconvergence on the circle of convergence of Taylor series with gaps [same J. **28** (1953), 197–203; MR **14**, 738] is extended to Laplace-Stieltjes transforms of the form $f(s)=\int_0^\infty e^{-su}dA(u)$ ($A(0)=0$, and $A(u) \in$ B.V. on each interval $[0, U]$). In this context, the role of gaps in the Taylor series is played by a sequence of intervals $|u-u_k| \leq \Delta_k$ in each of which the function $A(u)$ is constant. Let $\phi(x)$ denote the least concave majorant of $\log(|A(x)|+2)$, or alternatively, of $\log(\int_x^{x+1} |dA|+2)$, and suppose that $\int^\infty \phi(u)u^{-2}du < \infty$. If the function $f(\sigma+it)$ is bounded in the half-strip $\sigma>0$, $|t|<t_0$; if $f(it)=\lim_{\sigma\to 0+} f(\sigma+it)$ (p.p. in $|t|<t_0$), and, uniformly for $|t|<t_0$, $\int_0^\lambda \{f(iu)-A\}(\sin \lambda u/u)du \to 0$ as $\lambda\to\infty$; and if $\liminf_k \Delta_k/\phi(u_k) > c(t_0)$; then $A(u_k)\to A$. *G. Piranian* (Ann Arbor, Mich.).

Widder, David Vernon. Una trasformazione integrale connessa con la propagazione del calore. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 750–752.

A preliminary statement of the results proved in the paper reviewed below.

Widder, D. V. Integral transforms related to heat conduction. Ann. Mat. Pura Appl. (4) **42** (1956), 279–305.

Using properties of the Laplace transform, basic properties of the transform $f(t)=\int_0^\infty k(y, t)d\alpha(y)$, where $k(y, t)=\exp\{-y^2/4t\}/(4\pi t)^{\frac{1}{2}}$, are developed, with particular attention to the case in which $\alpha(y)$ is absolutely continous, so that $f(t)=\int_0^\infty k(y, t)\varphi(t)dt$. Then theorems concerning Cauchy's problem for the equation of heat con-

duction are obtained, and also results concerning the Poisson representation $u(x,t)=\int_{-\infty}^{\infty} k(x-y,t)d\alpha(y)$ of the temperature function are found. Finally relations are found between some of the integral transforms and operators such as $\exp\{xD^{\frac{1}{2}}\}$, where D is the differential operator. *D. L. Bernstein* (Los Angeles, Calif.).

Etienne, J. Transformation de Laplace des fonctions de plusieurs variables. Bull. Soc. Roy. Sci. Liège **25** (1956), 128–141.

The author obtains the usual theorems on region of convergence, transforms of linear combinations and convolutions, annihilation, uniqueness, and analyticity for the transform $F(p)=\int_{R_n} e^{-(p,x)}f(x)dx$, where $x=(x_1, \cdots, x_n)$ is a point in n-dimensional Euclidean space, $p=\xi+i\eta$, both ξ and η being in R_n, and $(p,x)=\sum_{k=1}^{n} p_k x_k$. Only absolutely convergent integrals are considered. There are also results concerning the behavior of $F(p)$ as p becomes infinite in certain directions. *D. L. Bernstein.*

Calderón, A. P.; and Zygmund, A. Algebras of certain singular operators. Amer. J. Math. **78** (1956), 310–320.

Let $x=(\xi_1, \cdots, \xi_n)$ be an element of Euclidean n-space, and $|x|=(\xi_1^2+\cdots+\xi_n^2)^{\frac{1}{2}}$. Let $K(x)$ be a homogeneous function of degree $-n$, such that $K(\lambda x)=\lambda^{-n}K(x)$ for every x and every $\lambda>0$, and let $\int K(x)d\sigma=0$, $\int |K(x)|^p d\sigma<\infty$, $p>1$, where the integrals are taken over the unit sphere $|x|=1$, and $d\sigma$ denotes the area-element. Then, according to the previous work of the authors [Acta Math. **88** (1952), 85–139; Amer. J. Math. **78** (1956), 289–309; MR **14**, 637, **18**, 894], for $f\in L^r$ one can consider

$$\tilde{f}_\varepsilon(x)=\int_{|x-y|>\varepsilon} K(x-y)f(y)dy,$$

and show that it converges pointwise almost everywhere, and in the mean of order r, as $\varepsilon\to 0$. Let the limit be denoted by $\tilde{f}$. The authors consider operators $\mathscr{T}(f)$ of the form $\mathscr{T}(f)=\alpha f+\tilde{f}$, α being a complex constant. Let $\mathscr{A}$ be the class of operators $\mathscr{T}$ with $K(x)$ in C^∞ for $|x|>0$, and $\mathscr{A}_p$ $(p>1)$ the class of operators $\mathscr{T}$ for which

$$(*) \qquad \|\mathscr{T}\|_p=|\alpha|+\Big[\int_{|x|=1}|K(x)|^p d\sigma\Big]^{1/p}<\infty.$$

It is proved that $\mathscr{A}$ is closed under addition and operator multiplication, while $\mathscr{A}_p$, with the norm (*), is a commutative, semi-simple Banach algebra, under operator multiplication. A Fourier transform of the operator $\mathscr{T}$ is defined, and some of its properties established. An operator in $\mathscr{A}$ (or $\mathscr{A}_p$) has an inverse in $\mathscr{A}$ (or $\mathscr{A}_p$) if and only if its Fourier transform does not vanish. The space of maximal ideals of $\mathscr{A}_p$ is homeomorphic to the sphere $|x|=1$. *K. Chandrasekharan* (Bombay).

See also: Combinatorial Analysis: Balliccioni. **Functions of Complex Variables:** Rabinovič. **Ordinary Differential Equations:** Doetsch. **Partial Differential Equations:** Ghizzetti. **Elasticity, Plasticity:** Muki.

Ordinary Differential Equations

★ **Stepanow, W. W. Lehrbuch der Differentialgleichungen.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1956. ix+470 pp.

This is the translation by J. Auth, D. Bebel, H. Friedrich from the sixth edition [Gostehizdat, Moscow, 1953] of the excellent Russian text by the late W. Stepanow. It presents in thorough detail standard material for what might constitute a second year graduate course in our universities, as it presupposes a certain maturity and acquaintance with analysis and more or less elementary algebra. The treatment of existence theorems is particularly extensive and there is a good introduction to the Lyapunov-Poincaré stability theory. Noteworthy also is an interesting historical chapter (the last). There are very many and varied problems with solutions given at the end of the book, and it terminates with an ample bibliography. *S. Lefschetz* (Princeton, N.J.).

★ **Poincaré, H. Les méthodes nouvelles de la mécanique céleste. Tome I. Solutions périodiques. Non-existence des intégrales uniformes. Solutions asymptotiques.** Dover Publications., Inc., New York, N. Y., 1957. v+382 pp. $2.75.

★ **Poincaré, H. Les méthodes nouvelles de la mécanique céleste. Tome II. Méthodes de MM. Newcomb, Gyldén, Lindstedt et Bohlin.** Dover Publications,Inc., New York, N. Y., 1957. viii+479 pp. $2.75.

★ **Poincaré, H. Les méthodes nouvelles de la mécanique céleste. Tome III. Invariants intégraux. Solutions périodiques du deuxième genre. Solutions doublement asymptotiques.** Dover Publications, Inc., New York, N. Y., 1957. ii+414 pp. $2.75.

The three volumes in this unaltered reprinting of Poincaré's monograph (1892, 1893, 1899) are paperbound. The price of the entire set is $7.50.

Beklemiševa, L. A. On the asymptotic behavior of solutions of certain non-linear systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 261–264. (Russian)

The author considers the behavior as $t\to+\infty$ of the solutions of $\ddot{x}+\sum_{k=1}^{s} b_k(1+o_k(t))t^{\alpha_k}x^{n_k}=0$, where the $o_k(t)$ satisfy inequalities of the form $|o_k(t)|<At^{-\varepsilon}$, $|o_k'(t)|<At^{-\varepsilon-1}$ for some $\varepsilon>0$; in order to avoid singularities, the n_k are to be positive rationals with odd denominators. First stated are necessary and sufficient conditions for all solutions to be continuable as $t\to+\infty$, namely, that n should have an odd numerator and $\bar{b}>0$ or $n\leq 1$; here $n=\max_k n_k$, and $\bar{b}$ denotes the value of b_k for that k-value, among the set for which $n_k=n$, for which α_k has the greatest value. A continuable solution does not increase more rapidly than some power of t, except when $n=1$, $\bar{b}<0$. An analogous pair of results deals with the exclusion of solutions for which x and $\dot{x}$ vanish together, and with the rate of decrease of other solutions. After these exclusions there remain for consideration "ordinary solutions", for which can be defined a "regime" ω, such that $\limsup_{t\to\infty}|x(t)|t^{-\omega+\varepsilon}=+\infty$, $\lim_{t\to\infty} x(t)t^{-\omega-\varepsilon}=0$ for any $\varepsilon>0$. A graphical procedure, too complicated to reproduce, is given to determine all possible regimes for a given equation; rules are stated giving the asymptotic behavior associated with regimes of various types. The method of proof is briefly indicated, and uses a change of variables together with two theorems on systems, which are also stated. The first form is a Hamiltonian form with perturbation terms, conditions being given for H to tend to a limit. The second form is $x_i''=\partial H(x,t)/\partial x_i+g_i(x,t)+h_i(x,x',t)$ $(i=1, \cdots, n)$, and conditions are given which ensure boundedness. *F. V. Atkinson* (Canberra).

Nehari, Zeev. Oscillation criteria for second-order linear differential equations. Trans. Amer. Math. Soc. **85** (1957), 428–445.

This paper concerns the differential equation (*) $y''+p(x)y=0$, where $p(x)\geqq 0$ is continuous on $0\leqq x<\infty$. The first part deals with necessary conditions that (*) be disconjugate for $x\geqq 0$, and centers around generalizations of results of Hille [same Trans. **64** (1948), 234–252; MR **10**, 376]. The necessary conditions are derived from a minimal principle [cf. Putnam, Duke Math. J. **16** (1945), 633–636; MR **11**, 437]. One of the necessary conditions derived is

$$x^{1-\beta}\int_0^x t^{\beta-\alpha-1}\Big(\int_t^\infty u^\alpha p(u)\,du\Big)\,dt\leqq \tfrac{1}{4}[1+(\beta-1)^{-1}(\alpha-1)^{-1}]$$
$$(0\leqq\alpha<1;\ \beta>1).$$

[For related results, see Hartman, Amer. J. Math. **73** (1951), 955–962, p. 957; **74** (1952), 389–400; MR **13**, 652; **14**, 50.] The second part concerns Sturm type comparison theorems involving inequalities between integrals of the coefficient functions. The third part deals with generalizations of Wiman's formula $\pi N(x)\sim\int_0^x p^{\frac{1}{2}}(x)dx$ for the number $N(x)$ of zeros of a solution of (*) on $(0, x)$ [see also Hartman, ibid. **73** (1951), 635–645, p. 642; MR **13**, 462].

P. Hartman (Los Angeles, Calif.).

Wittich, Hans. Defektfreie Lösungen linearer Differentialgleichungen. Arch. Math. **7** (1957), 459–464.

The problem is proposed to determine the linear differential equations of the form

$$(*)\qquad w^{(m)}+a_{m-1}(z)w^{(m-1)}+\cdots+a_1(z)w'+a_0(z)w=0,$$

where the a_j are polynomials, $a_0\not\equiv 0$, $m\geqq 2$, which have the property (E): There exists a fundamental system of solutions $w_1, \cdots, w_m$, each of which attains zero a finite number of times, while each solution w $(\not\equiv w_1, \cdots, w_m, 0)$ attains the value zero infinitely often and in such a manner that the Nevanlinna defect $\delta(w, 0)=0$. The answer is that a necessary and sufficient condition for an admitted differential equation to have the property (E) is that it be of the form

$$w''+tcw'-c^2w=0,$$

where c is a complex constant $\neq 0, \infty$ and t is a real constant. The proof is based on an analysis of the Wronskian of a fundamental system of the indicated type. Use is made of a Puiseux diagram associated with the differential equation and the following lemma. If $w_1, \cdots, w_m$ is a fundamental system of solutions of (*) which are transcendental and have respective orders $L_1, \cdots, L_m$, and if every solution of (*) which is not identically zero is transcendental, then $\sum L_j\geqq m$. Equality holds if and only if (*) has constant coefficients. If $w_1, \cdots, w_{m-1}$ of a fundamental system $(w_1, \cdots, w_m)$ are each transcendental functions of order one, then w_m is transcendental.

M. Heins (Providence, R.I.).

Babister, A. W. Response functions of linear systems with constant coefficients having one degree of freedom. Quart. J. Mech. Appl. Math. **10** (1957), 360–368.

The author considers the transient response for stable systems satisfying linear differential equations with constant coefficients and is interested in the accuracy with which the output follows the input. Criteria for optimizing the transient response are taken to be $L=\int_0^\infty e^2dt$ and $L_1=\int_0^\infty (de/dt)^2dt$, where e is the error at time t. A system whose parameters are chosen on the basis of minimizing L only leads to a slightly overdamped response often with a large undesirable overshoot. For a system with large overshoot, de/dt, and hence L_1, will be large; so the author chooses values of the parameters so that L is near its minimum value while L_1 is considerably lower than its value at $L_{\min}$. Such systems are less oscillatory. The quantities L and L_1 are expressed in terms of the coefficients of the characteristic equation and are observed to be quadratic in the initial conditions. There is also given a derivation of an integral formula for the response function L in terms of the frequency response spectrum of the system. This is based on the fact that the transient and the frequency response are related through a Fourier transformation. The response due to a step function, an initial impulse and a constant velocity input are each shown to be simply related to the response in free motion.

C. G. Maple (Ames, Iowa).

Pailloux, Henri. Opérateurs symboliques. Publ. Sci. Tech. Ministère de l'Air, Paris no. **317** (1956), iii+77 pp. 880 francs.

Basic properties of linear operators, and in particular operators related to the differential operator, are developed for one and then for several dimensions. Using the notion of generating function, examples are given to show how both ordinary and partial differential equations can be solved by operational methods; other illustrations, such as difference equations, are also given.

D. L. Bernstein (Los Angeles, Calif.).

Doetsch, Gustav. Das Anfangswertproblem für Systeme linearer Differentialgleichungen unter unzulässigen Anfangsbedingungen. Ann. Mat. Pura Appl. (4) **39** (1955), 25–37.

A system of linear differential equations with constant coefficients,

$$\sum_{\beta=1}^{N}\sum_{\nu=0}^{n} c_\nu^{\alpha\beta}\cdot Y_\beta^{(\nu)}=Y_\alpha(t)\quad(1\leqq\alpha\leqq N),$$

where $F_\alpha(t)$ are defined for $t>0$, has a solution satisfying the initial conditions $Y^{(\nu)}(+0)=Y_0^{(\nu)}$ $(0\leqq\nu\leqq n-1)$ at each point of continuity of the $F_\alpha(t)$, provided the determinant of coefficients of highest order, $|c_n^{\alpha\beta}|$, is not 0. One can easily find this by using the Laplace transformation. In case $|c_n^{\alpha\beta}|=0$, there will not be a solution, in general, but there are certain compatibility conditions which must be satisfied if a solution is to exist. When these compatibility conditions are not satisfied, it is still possible to obtain, by means of transforms, a solution of the equations if we use another set of initial values which do satisfy the compatibility conditions. The sense in which these are a "solution" is discussed, in terms of a particular problem from electric circuit theory. (It involves extending the domain of the problem to the entire t-axis.)

D. L. Bernstein (Los Angeles, Calif.).

Barrett, John H. A Prüfer transformation for matrix differential equations. Proc. Amer. Math. Soc. **8** (1957), 510–518.

The author considers the linear second-order matrix differential equation

$$(I)\qquad (P(x)Y')'+F(x)Y=0\quad(a\leqq x<\infty)$$

in an $n\times n$ matrix $Y(x)$, where $P(x)$, $F(x)$ are $n\times n$ symmetric matrices of real-valued continuous functions with $P(x)$ positive definite on $a\leqq x<\infty$. Analogous to the Prüfer polar-coordinate transformation for a real self-adjoint linear homogeneous scalar differential equation of

the second order, it is shown that if $Y(x)$ is a solution of (I) satisfying $Y(a)=0$, and $Y(x)$ is not identically singular on $a\leqq x<\infty$, then there is a non-singular $R(x)$ and a real symmetric $n\times n$ matrix $Q(x)$ such that $Y(x)=S^*(x)R(x)$, $P(x)Y'(x)=C^*(x)R(x)$, where $S(x)$, $C(x)$ is the solution of the matrix differential system

(II) $S'=Q(x)C,\ C'=-Q(x)S,\ S(a)=0,\ C(a)=I.$

The last portion of the paper is devoted to certain non-oscillation theorems for (I) and (II). In Theorem 2.1 and Corollary 3.2.1 of the paper the condition that $Y(x)$ is a "non-trivial" solution of the above equation (I) should be interpreted as meaning that $Y(x)$ is not identically singular. It is to be commented that the author does not state specifically that the coefficient matrices of the considered equations are real-valued, and leaves this to be inferred from the use of the word "real" in Lemma 1.1 and such phrases as "P is symmetric and positive definite"; although the proofs of some of the results of the paper are true without this assumption of reality, and all results of the paper may be generalized to self-adjoint equations with complex coefficients, the presented proofs of the central results of the paper do require this assumption. *W. T. Reid* (Los Angeles, Calif.).

Wintner, Aurel. On stable oscillations of high frequency. Boll. Un. Mat. Ital. (3) **12** (1957), 9–11.

It is well known that, if a continuous function, $\omega(t)$, defined for $t\geqq t_0$, tends to ∞ as $t\to\infty$, then all solutions of (*) $d^2x/dt^2+\omega^2(t)x=0$ stay bounded as $t\to\infty$, provided that $d\omega(t)\geqq 0$, $t\geqq t_0$. This is not in general true if the latter condition is omitted [A. Wintner, J. Appl. Phys. **18** (1947), 941–942; MR **9**, 285]. The author shows that if $Q=\{t|d\omega(t)\geqq 0\}$, $R=\{t|d\omega(t)<0\}$, $\omega(t)\geqq 0$, $t\geqq t_0$, $\omega(t)\to\infty$ as $t\to\infty$, $t\in Q$, $\int_R |d\log\omega(t)|<\infty$, then all of the solutions of (*) remain bounded as $t\to\infty$. *J. K. Hale.*

Barbuti, Ugo. Sulla nozione di t_∞-similitudine tra matrici e sulla stabilità dei sistemi differenziali lineari. Boll. Un. Mat. Ital. (3) **12** (1957), 61–66.

The matrices $A(t)$ and $B(t)$ are said to be t_∞-similar [R. Conti, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 247–250; MR **18**, 483] if the system (*) $x'=A(t)x$ can be transformed, by $x=T(t)Z$, into $Z'=(B(t)+C(t))Z$ with $\int_0^\infty \|C(t)\|dt<\infty$. It is known that t_∞-similarity preserves both uniform and restricted stability. Generalizing a result of Cesari [Ann. Scuola Norm. Sup. Pisa (2) **9** (1940), 163–186; MR **3**, 41], the author proves the theorem: If $A(t)$ is of bounded variation in $[0,\infty)$, summable in every finite interval of $[0,\infty)$, and $A(t)\to L$ as $t\to\infty$, where L is a constant matrix with simple characteristic roots, then $A(t)$ is t_∞-similar to its canonical form. An interesting example is given to show that, if $A(t)$ is not t_∞-similar to its canonical form, then the stability of (*) cannot be decided by studying the canonical differential system. However, for a general class of matrices $A(t)$, the author shows how to construct a diagonal matrix which is t_∞-similar to $A(t)$. *J. K. Hale.*

Eršov, B. A. A method of approximate estimation in investigations of stability of non-linear control systems. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 22–27. (Russian)

The author obtains an estimate of the growth of solutions of the differential system $dX/dt=Ax+F(X)$, where A is a constant $n\times n$ matrix, and X and $F(X)$ are column vectors. The roots of the characteristic equation $D(\lambda)=|A-\lambda I|=0$ are assumed to be simple and have negative real parts, and the ith component of $F(X)$ is assumed to be of the form $h_i f(x_1,\cdots,x_n)$, where h_i is a constant and f satisfies the conditions $f(0,\cdots,0)=0$ and $|f(x_1,\cdots,x_n)|\leqq\alpha\sum_i|x_i|$. *L. A. Zadeh.*

Skatchkov, B. N. Questions of stability in the large and regulating properties for a certain system of differential equations. I. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1957), no. 3, 67–80. (Russian. English summary)

A certain regulating system is represented in suitable coordinates by

$$(*)\qquad \begin{aligned}\dot\eta_1&=r_1\eta_1+n_1\xi\\ \dot\eta_2&=r_2\eta_2+n_2\xi\\ \dot\xi&=f(\eta_1,\eta_2,\xi)\end{aligned}$$

where $r_i<0$, n_j is a constant, and if $\sigma=p_1\eta_1+p_2\eta_2-\xi$, p_k real constant, then $f=0$ for $\sigma=0$, f being an arbitrary continuous function such that the solutions of (*) are unique on $\sigma=0$. The author shows, by examining various special cases and using at times Lyapunov's second method, that if $1+|n_1p_1|/r_1+|n_2p_2|/r_2>0$ then the system (*) is stable in the whole space. *S. Lefschetz.*

Bogdanov, Yu. S. Characteristic numbers of systems of linear differential equations. Mat. Sb. N.S. **41**(83) (1957), 481–498. (Russian)

Let the system (1) $\dot x_i=\sum p_{ij}x_j$ $(i,j=1,2,\cdots,n)$ have coefficients continuous and bounded for $t\geqq t_0$. Given a set of solutions of (1), let S be the sum of its characteristic numbers (c.n.) in the sense of Lyapunov (whose definitions are followed), let $L=$c.n. $\exp(-\sum\int_{t_0}^t p_{ii}dt)$, and set $\sigma=-S-L$. Take now a normal system of solutions and let its c.n.'s be λ_i, ranged so that $\lambda_1\geqq\lambda_2>\cdots\geqq\lambda_n$. Similarly take a normal system of solutions of the adjoint to (1) with c.n.'s λ_i' such that $\lambda_1'\leqq\lambda_2'\leqq\cdots\leqq\lambda_n'$. Then $0>\lambda_i+\lambda_i'\geqq-\sigma$. Furthermore,

$$\sigma\leqq\limsup_{t\to+\infty}\frac{n}{2t}\int_{t_0}^t\sum|p_{ij}+p_{ji}|dt-\liminf_{t\to+\infty}\frac{1}{t}\int_1^t\sum p_{ii}dt.$$

Comparing, also, (1) and (2) $\dot y_i=\sum(p_{ji}+q_{ji})y_i$, the following result is proved. If the c.n.'s of the q_{ji} exceed σ then (1) and (2) have the same c.n.'s. *S. Lefschetz.*

Vinograd, R. È. Inapplicability of the method of characteristic exponents to the study of non-linear differential equations. Mat. Sb. N.S. **41**(83) (1957), 431–438. (Russian)

Perron's order number, $\lambda(x)$, of any solution $x(t)\not\equiv 0$ of an n-vector equation $\dot x=F(t,x)$, where $F(t,x)$ is everywhere continuous and lipschitzian in x for a constant $K>0$ and $F(t,0)\equiv 0$, satisfies $|\lambda(x)|\leqq K$. Let $\Lambda=\sup\lambda(x)$ over all non-trivial solutions. The author proves that $\Lambda<0$ alone does not imply the asymptotic stability of $x=0$ as it does when F is linear in x. As an example, he discusses in detail the system

$$\dot x\varphi(x,y)=x^2(y-x)+y^5,\ \dot y\varphi(x,y)=y^2(y-2x),\qquad \varphi(x,y)=(x^2+y^2)[1+(x^2+y^2)^2]$$

for which $\Lambda\leqq-1/15$ and $x=0$ is unstable. *H. A. Antosiewicz* (Providence, R.I.).

Vinograd, R. È. On the central characteristic index of a system of differential equations. Mat. Sb. N.S. **42**(84) (1957), 207–222. (Russian)

Continuing his study of Perron's order numbers of the solutions of non-linear n-vector equations (1) $\dot x=F(t,x)$

[see the paper reviewed above], the author first considers equations (2) $\dot{x}=A(t)x$ with $\sup|A(t)|<\infty$ on $[0,\infty)$, for which he defines a "central characteristic number Ω" as follows. Let $X(t)$ be a base for the solutions of (2) and let Φ be the class of functions φ (shown to be non-empty) such that $|X(t)X^{-1}(\tau)|\leq C\exp\int_\tau^t\varphi ds$ for $\varphi\in\Phi$ and $0\leq\tau<t<\infty$, where the constant C may depend upon φ; then $\Omega=\inf_{\varphi\in\Phi}\lambda(\int_0^t\varphi ds)$ where λ is Perron's order number. Among other results, the author then proves these theorems: (i) For any $\varepsilon>0$ there is a $\delta>0$ such that every solution $x(t)$ of (1), with $F(t,x)\equiv A(t)x+f(t,x)$ and $|f(t,x)|\leq\delta|x|$ for small $|x|$, satisfies $|x(t)|\leq|x(0)|B(\varepsilon)\exp(\Omega+\varepsilon)t$ for $t\geq0$. (ii) If $\Omega<0$ and if $F(t,x)\equiv A(t)x+f(t,x)$ with $|f(t,x)|\leq K|x|^{1+\alpha}$ $(\alpha>0, K>0)$ for small $|x|$, then the trivial solution $x(t)\equiv0$ of (1) is asymptotically stable.

H. A. Antosiewicz (Providence, R.I.).

Gillies, A. W. On a class of differential equation governing non-linear vibrations. Quart. J. Mech. Appl. Math. **10** (1957), 342–359.

The author makes use of the method previously applied to particular second- and third-order equations [same J. **8** (1955), 107–128; **10** (1957), 101–121; MR **16**, 926; **19**, 142] to determine periodic solutions of the equation of the form $f(D)x+g(D)y=F(t)$. Here, D is the operator d/dt, y is a function of x, expandible in a series in powers of x, and $F(t)=2B\cos\omega t$. The approximate linear equation when x and y are small is assumed to have one nearly sinusoidal normal mode, its other modes being exponentially damped with time constants less than a constant which is $O(1)$.

M. Zlámal (Brno).

Taam, Choy-Tak. On the solutions of nonlinear differential equations. I. J. Math. Mech. **6** (1957), 287–300.

The author gives results of several kinds concerning the solutions of

$$(*)\qquad x''+p(t)x+2q(t)x^3=0,$$

where $p(t)$ and $q(t)$ satisfies restrictions of Lebesgue type and inequalities of the form $0<p_1\leq p(t)\leq p_2$, $0<q_1\leq q(t)\leq q_2$. Considering the first positive zero T_x and the first maximum of the solution such that $x(0)=0$, $x'(0)=E>0$, inequalities are found connecting these with similar quantities for $u''+p_1u+2q_1u^3=0$, $v''+p_2v+2q_2v^3=0$; the limiting behaviour of T_x as $E\to0$, $E\to\infty$ is also found. Other comparison theorems relate to monotonic $p(t)$, $q(t)$; e.g., if $p(t)$, $q(t)$ are non-decreasing, the distance between successive zeros is non-increasing. Next follow two results for the case when $p(t)$, $q(t)$ have a least common period L, which assert the existence of solutions of period nL with specified numbers of zeros in a period. Finally, if $p(t)$, $q(t)$ are of bounded variation, it is noted that the solutions of (*) are bounded; under more drastic restrictions, an asymptotic approximation holds for solutions of (*) in terms of the solutions of $\phi''+\phi+2\phi^3=0$. The extension of these results from (*) to

$$(mx')'+\sum_{k=1}^{n}kp_kx^{2k-1}=0$$

is discussed.

F. V. Atkinson (Canberra).

Johnson, Elgy S.; and Taam, Choy-Tak. On the solutions of nonlinear differential equations. II. J. Math. Mech. **6** (1957), 383–392.

Let $x=x(t,E)$ be the solution with $x(0)=0$, $x'(0)=E>0$, of $x''+p(t)x-2q(t)x^3=0$, where $p(t)$, $q(t)$ are measurable and have positive lower and upper bounds for $t\geq0$. It is proved that there exists $E_0>0$ with these properties: (i) For $0<E<E_0$, $x(t,E)$ has a first positive zero at $t=T(E)$ and a positive maximum $M(E)$ in $(0,T(E))$, first attained at $t=T'(E)$; $T(E)$, $M(E)$, $T'(E)$ are increasing functions of E. As $E\downarrow0$, $M(E)\to0$ and $T(E)\to T_0$, where $t=T_0$ is the first positive zero of the solution $w(t)$ of $w''+p(t)w=0$ with $w(0)=0$, $w'(0)=1$. As $E\uparrow E_0$, $T(E)\to+\infty$ and $M(E)$ tends to a finite limit. (ii) $x(t,E_0)$ exists and is bounded for $t\geq0$, and has no positive zeros. (iii) For $E>E_0$, $x(t,E)\to+\infty$ as t tends to some finite limit $I(E)$; $I(E)$ is a decreasing function of E and tends to $+\infty$ as $E\downarrow E_0$, to 0 as $E\to+\infty$. If $p(t)$, $q(t)$ are even periodic functions with the common least period L, the above results can be used to prove: (iv) If n is a positive integer with $nL>2T_0$, then some $x(t,E)$ is periodic with least period nL and has exactly three zeros on $[0,nL]$. (v) If $a=\inf p(t)$ and m, n are positive integers with $nLa^{\frac12}>2m\pi$, then some $x(t,E)$ is periodic with least period nL and has exactly $(2m+1)$ zeros on $[0,nL]$. Finally some oscillation and comparison theorems are proved.

G. E. H. Reuter (Manchester).

Biglov, Z. I. Expansion according to eigenfunctions of a system of differential equations of second order. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 797–799. (Russian)

The expansion theorem is given for the system $l(y)=-y''+P(x)y$ on $0\leq x<\infty$, where $P(x)$ is a real symmetric matrix integrable on the interval $(0,\infty)$ and y is a vector with n components. The boundary condition at $x=0$ is $y'(0)=\theta y(0)$, where θ is a hermitian matrix. Because of the integrability of $P(x)$, rather explicit formulas are given for the expansion theorem, the resolvent operator, etc., generalizing the familiar case where y is a scalar.

N. Levinson (Cambridge, Mass.).

★ **Sansone, Giovanni; e Conti, Roberto. Curve caratteristiche di sistemi omogenei.** Scritti matematici in onore di Filippo Sibirani, pp. 243–260. Cesare Zuffi, Bologna, 1957.

Systems $\dot{x}=X_m(x,y)$, $\dot{y}=Y_m(x,y)$, where X_m, Y_m are sufficiently regular homogeneous functions of degree m, are considered. The qualitative behaviour of the solutions may be completely described by means of simple criteria.

J. L. Massera (Montevideo).

Dolaptschiew, Bl.; und Tschobanow, Iw. Über die Halmsche Differentialgleichung. Z. Angew. Math. Mech. **37** (1957), 233–235.

Hebroni, Pessach. On L-functions in the abstract differential ring with application to integrodifferential equations. I, II. Riveon Lematematika **9** (1955), 54–69; **10** (1956), 49–67. (Hebrew)

Let R be a non-commutative real Banach algebra with a derivation $a\to a'$ defined on a subring R_d of R and an integration $a\to\int a$ defined for all $a\in R$, with the relation $\int(a')=a+c$ for some constant c (initial value). The author has shown in an earlier paper [Compositio Math. **5** (1938), 403–429, p.420] that the solutions of the differential equation $y'=cy$, c a constant, have the form $y=\sum(n!)^{-1}(\zeta c)^n$ where $\zeta=\int1$. In these two papers he extends this result; he considers the equation (*) $y'=f(y)$, where f is a linear mapping of R_d into itself satisfying $f(\zeta y)=\zeta f(y)$, $f(a')=f(a)'$, $|f(y)|\leq H|y|$ for some positive number H. A typical example of an L-function is $f(y)=\sum a_iyb_i$, where a_i and b_i are constants for which the series $\sum|a_i||b_i|$ is convergent.

To a series $\sum_{\nu}^{\infty} \alpha_n z^{n-\nu}$ which represents an integral function $F(z)$ the author makes correspond a well-defined function (L-series) on $R_{\mathfrak{d}}$; $C_F=\sum_{\nu}^{\infty} \alpha_n \zeta^{n-\nu} f_n(a)$, where $f_0(a)=a$, $f_n(a)=f(f_{n-1}(a))$. It is shown, among other things, that $C_{FG}(a)=C_F(C_G(a))$. The main result is that all solutions of (*) have the form $y=\sum (n!)^{-1}\zeta^n f_n(c)=E(c)$ for arbitrary constants c. Moreover, if I is a closed ideal in R and $f(I)\subset I$, then the solutions of (*) are $y=E(c)$ with constants $c \in I$. By similar methods the solutions of the nonhomogeneous equation $y'=f(y)+b$ (b a constant satisfying a certain condition) are completely determined. The papers include also various applications to rings of infinite matrices and to Banach algebras of the type dealt with in the paper quoted above, which lead to certain integro-differential equations. *S. A. Amitsur.*

See also: Fields, Rings: Goldman. **Functions of Real Variables:** Sternberg and Wintner. **Functions of Complex Variables:** Levin. **Partial Differential Equations:** Hornich; Eringen. **Integral and Integrodifferential Equations:** Melzak. **Mechanics of Particles and Systems:** Rund. **Optics, Electromagnetic Theory, Circuits:** Jauch. **Quantum Mechanics:** Kato.

Partial Differential Equations

★ **Séminaire Schwartz de la Faculté des Sciences de Paris, 3e année: 1955/1956. Problèmes mixtes pour l'équation des ondes.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+30 pp. (polycopiées)

Ce Séminaire se compose de 7 exposés sur les problèmes mixtes (au sens de Hadamard précisé en théorie des distributions) pour l'équation $\Box u=g$, où $\Box=\partial^2/\partial t^2-\Delta$. L'avantage essentiel de ces exposés est de donner un aperçu assez efficace de l'emploi des distributions dans les problèmes considérés.

Les cinq premiers exposés concernent la méthode de la transformation de Laplace par rapport à t, d'après le point de vue de Schwartz et ses élèves [voir Lions, Acta Math. **94** (1955), 13–153; MR **17**, 745]. On est alors amené à l'étude préalable de l'opérateur $D_p=p^2-\Delta$, p complexe; cette étude a été faite, avec λ au lieu de p^2, dans le même Séminaire [1954–55; MR **19**, 279]; on la résume ici (exposé n. 1): Soit $\mathfrak{E}_k$ ($k\geqq 0$) l'espace hilbertien des distributions f sur un ouvert Ω de R^n, telles que $D^m f \in L^2(\Omega)$ pour $0\leqq|m|\leqq k$, le produit scalaire étant défini sur $\mathfrak{E}_k$ par

$$(*) \qquad (f, g))_k=\sum_{|p|\leqq k}\int_{\Omega} D^p f(x) D^p \overline{g(x)}\,dx.$$

On désigne par $\mathfrak{D}_k$ le complété de l'espace $\mathfrak{D}(\Omega)$ (des fonctions indéfiniment dérivables à support borné contenu dans Ω) pour la norme associée au produit (*); et par $\mathfrak{D}_k'$ ou $\mathfrak{D}_{-k}$ le dual de $\mathfrak{D}_k$, "réalisé" dans $\mathfrak{D}'(\Omega)$. Or, pour tout p tel que $\Re p>0$, l'opérateur D_p est un isomorphisme topologique de $\mathfrak{D}_1$ sur $\mathfrak{D}_1'$, et son inverse, $\mathfrak{G}_p$, est un "opérateur de Green", qui permet de résoudre $D_p u=T$, $T \in \mathfrak{D}_1'$, avec la "condition au contour" $u-f \in \mathfrak{D}_1$ (c'est-à-dire: $u-f$ "nulle au contour"), f étant donnée dans $\mathfrak{E}_1$. En outre $\mathfrak{G}_p$ est, pour $\Re p>0$, une fonction holomorphe de p à valeurs dans $\mathfrak{L}(\mathfrak{D}_1', \mathfrak{D}_1)$, telle que $\|\mathfrak{G}_p\|\leqq k|p|$, k constant. L'image de cette fonction de p par la transformation inverse de Laplace est donc une distribution $G(t)$, à valeurs dans $\mathfrak{L}(\mathfrak{D}_1', \mathfrak{D}_1)$, nulle pour $t<0$. (Dans l'exposé n. 2 on rappelle quelques notions fondamentales sur les distributions vectorielles et sur la transformation de Laplace de telles distributions.) Le "problème de Cauchy grossier" (exposé n. 4) consiste à résoudre $\Box x=A$ dans $\mathfrak{D}_t'+(\mathfrak{D}_1)$ (espace des distributions de support limité à gauche et à valeurs dans $\mathfrak{D}_1$); on a $X=G *_t A$. Dans les "problèmes de Cauchy fins", on cherche des conditions qu'il faut imposer à A pour que X ait la régularité désirée, par rapport aux conditions initiales (d'une part) et aux conditions au contour (d'autre part); à cet effet on considère $G(t)$ comme distribution à valeurs dans un espace $\mathfrak{L}(E, F)$, où F est plus large que $\mathfrak{D}_1$ ou bien E est plus restreint que $\mathfrak{D}_1'$. Dans l'exposé n. 5 on fait une discussion détaillée de ces problèmes.

Les deux derniers exposés concernent la méthode généralisée des ondes stationnaires. On cherche alors s et U tels que $\Delta u+s^2u=0$, avec $U \in \mathfrak{D}_1$ (i.e. "nulle au bord"). Si Ω est borné, il existe une suite de nombres $0<s_1\leqq\cdots\leqq s_\nu\leqq\cdots$ et une suite de fonctions $U_\nu \in \mathfrak{D}_1$ vérifiant $\Delta u_\nu+s_\nu^2 u=0$, et $\{U_\nu\}$ est une suite orthogonale complète, par exemple, dans $\mathfrak{D}_1'$. Dans l'exposé n. 7, on esquisse la résolution de $\Box u=g$, avec $u(0)=u'(0)=0$, où $g(t)$ est une fonction de t à valeurs dans $\mathfrak{D}_k'$ ($k\geqq -1$) et l'inconnue $u(t)$, une fonction à valeurs dans un espace de distributions sur Ω. Pour la résolution, on utilise un développement (non unique si $k\geqq 2$), $g(t)=\sum g_\nu(t)V_\nu(t)$, où $V_\nu=(1+s_\nu^2)^k U_\nu$, et on cherche un développement analogue pour $u(t)$.

J. Sebastião e Silva (Lisbonne).

★ **Hille, Einar. Some aspect of Cauchy's problem.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 109–116. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The abstract Cauchy problem, formulated by the author and R. S. Phillips reads: Given a complex B-space X and a linear operator U with domain $D(U)$ and range $R(U)$ in X and given $y_0 \in X$, find a solution $y(t)=y(t, y_0)$ such that $y(t)$ is strongly continuously differentiable in each finite subinterval of $[0, \infty)$ (or $(0, \infty)$) and $y'(t)=Uy(t)$ (for $t>0$), and $\lim_{t\downarrow 0}\|y(t, y_0)-y_0\|=0$. The corresponding problem is denoted by ACP_1^1 (or ACP_2^1), and the problem relating to the higher order equation $y^{(n)}(t)=U^n y(t)$ is denoted by ACP^n. A number of important results, due to the author, Phillips and W. Feller are discussed. Theorem 1 (Hille): If U is a closed operator whose point spectrum is not dense in any right half-plane, then for each $y_0 \in X$ the ACP_1^1 has at most one solution of normal type, i.e., solution of exponential growth at $t=\infty$. Theorem 2 (Hille): The non-uniqueness of the solution of normal type is, when U is closed, equivalent to the existence of a non-trivial solution of $Ux(\lambda)=\lambda x(\lambda)$, which is bounded and holomorphic in each half-plane $\Re(\lambda)\geqq\varepsilon+\limsup_{t\to\infty} t^{-1}\log\|y(t)\|$, $\varepsilon>0$. The author has discovered: If the point spectrum of U covers the whole complex plane, then an "explosive" solution may exist for which $\lim_{t\to t_0}\|y(t)\|=\infty$ for $t_0<\infty$. Theorem 3 (Phillips): If U is the infinitesimal generator of a semi-group $S(t)$ of Phillips' class $(0, A)$, then the ACP_2^1 has a unique solution $y(t, y_0)=S(t)y_0$ for each $y_0 \in D(U)$. Theorem 4 (Phillips) gives a converse to Theorem 3. For the diffusion operator $C(S)=b(x)S_{xx}+a(x)S_x=S_t$ and its adjoint $L(T)=(b(x)T)_{xx}-(a(x)T)_x=T_t$ with coefficients $a(x)$ and $b(x)>0$, continuous in (α, β), $-\infty\leqq\alpha<0<\beta\leqq\infty$, Theorem 5 (Hille and Feller) states: The Cauchy problem for $C(S)$ in $C[\alpha, \beta]$ has a unique solution for every $y_0 \in D(C)$ if and only if $\Omega(\alpha)=\Omega(\beta)=\infty$. The Cauchy problem for $L(T)$ in $L(\alpha, \beta)$ has a unique solution for every $y_0 \in D(L)$ if and only if $W_2(\alpha)=W_2(\beta)=\infty$. Here:

$$\Omega(x)=\int_0^x (W_1(x)-W_1(s))dq(s),\quad W_2(x)=\int_0^x W_1(s)dq(s),$$

$$W_1(x)=\int_0^x W(s)ds,\qquad q(x)=\int_0^x (b(s)W(s))^{-1}ds,$$

$$W(x)=\exp(-\int_0^x a(s)b(s)^{-1}ds).$$

Fellers' classification of the boundary into the regular-, exit-, entrance- and natural-boundary corresponds to: (reg.) $\Omega(\beta)<\infty$, $W_2(\beta)<\infty$; (exit) $\Omega(\beta)<\infty$, $W_2(\beta)=\infty$; (ent.) $\Omega(\beta)=\infty$, $W_2(\beta)<\infty$; and (nat.) $\Omega(\beta)=\infty$, $W_2(\beta)=\infty$. The final section of the paper is devoted to ACP^n, in particular, a theorem (Hille), expressing the solution in Stokes' form, is given for the case $n=2$ and where U generates a group. *K. Yosida* (Tokyo).

Lions, J. L. Solutions élémentaires de certains opérateurs différentiels à coefficients variables. J. Math. Pures Appl. (9) **36** (1957), 57–64.

Malgrange [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355; MR **19**, 280] and others have shown the existence of "elementary solutions" E for partial differential operators D with constant coefficients; here, E is a distribution, and $DE=\delta$. In the present note, the author shows how this can be extended to a class of operators with variable coefficients. This depends upon the following special result dealing with equivalence of operators. If $L=d^2/dx^2+r(x)d/dx+s(x)$, with $r(x)$ and $s(x)$ of class C^∞, then there is an automorphism U of $\mathscr{D}'$ such that $ULU^{-1}=d^2/dx^2$. An elementary solution for d^2/dx^2 then yields one for L. This is then immediately extended to a class of partial differential operators formed from products of such ordinary operators L. In a recent announcement [Delsarte and Lions, C. R. Acad. Sci. Paris **244** (1957), 832–834; MR **19**, 273], this equivalence theorem has been extended to nth order ordinary operators L having analytic coefficients. This at once permits a similar treatment of a larger class of partial differential operators D with variable (analytic) coefficients.

R. C. Buck (Madison, Wis.).

★**Legras, J. Techniques de résolution des équations aux dérivées partielles: équation de la chaleur, équation de Laplace, équation des ondes.** Dunod, Paris, 1956. xv+180 pp. 1450 francs.

As the title indicates, this small book is a discussion of certain simple classical techniques which can be used to get solutions of certain standard boundary value problems, for the heat equation, Laplace's equation, and the wave equation, in 2 dimensions. There is also a description of simple numerical methods which can be used in certain cases. Some of the problems are also discussed, in the last chapter, for 3 and 4 independent variables. In setting up the boundary value problems, the related physical situation is always mentioned. *D. L. Bernstein.*

Princivalli, M. L. Su un teorema di unicità per un problema al contorno relativo all'equilibrio delle volte cilindriche. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 235–245 (1956).

The author establishes a uniqueness theorem for a system of equations considered in a previous paper [same Ann. (3) **8** (1954), 157–291; MR **17**, 627] but under more general hypotheses. Specifically, she proves that every solution $u=(u_1, u_2, u_3)$ of the following system of equations:

$$\Delta_2 u_1+k\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+\frac{\partial}{\partial x}(bu_3)=0,$$

$$\Delta_2 u_2+k\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+\frac{\partial}{\partial y}(cu_3)=0,$$

$$b\frac{\partial u_1}{\partial x}+c\frac{\partial u_2}{\partial y}+\Delta_4 u_3+au_3=0,$$

which satisfies the conditions $u_1=u_2=u_3=\partial u_3/\partial \nu=0$ on the boundary of a region A, vanishes identically in A. Here k is a constant greater than $-\frac{1}{2}$ and a, b, c are functions of x and y which satisfy rather general hypotheses.

D. L. Bernstein (Los Angeles, Calif.).

Saltykow, Nicolas. Équations aux dérivées partielles intégrables par séparation des variables. C. R. Acad. Sci. Paris **242** (1956), 2090–2093.

A short discussion is given of how one can determine all the equations of the form $F(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n)=0$, where $p_i=\partial z/\partial x_i$, which satisfy the following condition for integration by separation of the variables:

$$D\left(\frac{F, F_k/F_{n+k}}{x_i, p_i}\right)=0,$$

where $F_k=\partial F/\partial x_k$, $F_{n+k}=\partial F/\partial p_k$, and $1\leqq i\leqq n$, $i\neq k$. If one sets $H=\sum_{i,k} A_{ik}p_ip_k+U$, then the above equations can be replaced by the system

$$\text{(I)}\qquad \frac{\partial \varphi_r}{\partial x_i}\frac{\partial H}{\partial p_i}-\frac{\partial \varphi_r}{\partial p_i}\frac{\partial H}{\partial x_i}=0\quad (r+1\leqq i\leqq n;\ 1\leqq r\leqq n-1),$$

$$\text{(II)}\qquad \frac{\partial H}{\partial x_k}-\varphi_k\frac{\partial H}{\partial p_k}=0\quad (1\leqq k\leqq n),$$

where $\varphi_k=F_k/F_{n+k}$. Then one can study the singular solutions of this system, and use this to obtain the number of types of equations integrable by separation of variables. If this number for n independent variables is denoted by Σ_n, then it turns out that $\Sigma_n=2^{n(n-1)/2}$.

D. L. Bernstein (Los Angeles, Calif.).

Pliś, A. A method of determining the existence domain for solutions of partial differential equations of the first order. Ann. Polon. Math. **3** (1957), 183–188.

The classical characteristic equations for the differential equation

$$\text{(1)}\qquad \frac{\partial z}{\partial x}=f\left(x, y_1, \cdots, y_n, z, \frac{\partial z}{\partial y_1}, \cdots, \frac{\partial z}{\partial y_n}\right)$$

are, on setting $Y=(y_1, \cdots, y_n)$ and $Q=(q_1, \cdots, q_n)$,

$$\text{(2)}\quad y_i'=-f_{q_i}(x, Y, z, Q);\quad z'=f-\sum_{k=1}^n q_kf_{q_k};\quad q_i'=f_{y_i}+q_if_z\quad (1\leqq i\leqq n).$$

Assume that $f(x, Y, z, Q)$ is of class C^1 and has continuous second derivatives with respect to all variables except x; assume that (2) has a solution on $J=(a_1, a_2)$ of the form (3) $Y=\Phi(x)$; $z=\zeta(x)$; $Q=\Gamma(x)=(\gamma_1(x), \cdots, \gamma_n(x))$. If a solution $z(x, Y)$ of (1) and the initial condition $z(a, Y)=\omega(Y)$, where $a\in J$, is of class C^2 in a neighborhood of the arc $Y=\Phi(x)$, then the functions $t_{ij}=z_{y_iy_j}(x, \Phi(x))$ $(1\leqq i, j\leqq n)$ satisfy a certain system of supplementary characteristic equations (I) (which are quite lengthy and so are omitted here). Furthermore, if $Y=Y(x, V)$, $z=s(x, V)$, $Q=Q(x, V)$ denote the solution of (2) passing through $[V, \omega(V), \omega_{y_i}(V)]$ for $x=a$, for any V in the neighborhood of $\Phi(a)=H$, then the functions $u_i^j(x)=\partial y_i(x, H)/\partial v_j$ have non-vanishing determinant $|u_i^j(x)|$ on J.

Conversely, if a solution $t_{ij}=\tau_{ij}(x)$ of equations (I) is defined on J, and if $\omega(Y)$ is of class C^2 in a neighborhood of H, where $\omega(H)=\zeta(a)$, $\omega_{y_i}(H)=\gamma_i(a)$, $\omega_{y_iy_j}(H)=\tau_{ij}(a)$, then there exists a function $z(x, Y)$ of class C^2 in a neighborhood of the arc $Y=\Phi(x)$, for $x \in J$, satisfying (1) and (3). Also, $z_{y_iy_j}(x, \Phi(x))=\tau_{ij}(x)$. *D. L. Bernstein.*

Hornich, Hans. Über die nirgends lösbaren linearen partiellen Differentialgleichungen. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 103–109.

The following theorem is proved. Given a continuous function $f(x, y)$ with f_y continuous in a region B, and given a function $\delta(\rho)$, defined for all non-negative ρ, such that $\delta(0)=0$, $\delta(\rho)>0$ for $\rho>0$, $\delta(\rho_1)+\delta(\rho_2)\geqq\delta(\rho_1+\rho_2)$, $\delta(\rho_2)\geqq\delta(\rho_1)$ for $\rho_2>\rho_1$, $\int_0 d\lambda/\delta(\lambda)<+\infty$, there exists a function $\phi(x, y)$ defined on B such that $|\phi(p_1)-\phi(p_2)|\leqq c\delta(p_1p_2)$ and the differential equation $u_x+\phi(x, y)u_y=f(x, y)$ is non-solvable in every subregion G of B in which f_y does not vanish identically. This result generalizes one previously established by the author, in which ϕ satisfies a Hölder condition [Monatsh. Math. **59** (1955), 34–42; MR **16**, 825]. *D. L. Bernstein* (Los Angeles, Calif.).

Hornich, Hans. Existenzsätze bei gewöhnlichen und partiellen Differentialgleichungen und zugehörige metrische Geometrie. Comment. Math. Helv. **31** (1956), 108–110.

The previous results on non-solvability of the equation $u_x+\phi(x, y)u_y=f(x, y)$ established by the author are contrasted with the corresponding situation in ordinary differential equations, by describing the geometric pictures associated with possible solutions, using as a metric the function $\delta(\rho)$ previously introduced. [See Hornich, Monatsh. Math. **59** (1955), 34–42; MR **16**, 825; see also the paper reviewed above.] *D. L. Bernstein.*

Cinquini, Silvio. Sopra l'unicità della soluzione dei sistemi di equazioni a derivate parziali del primo ordine. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 960–978.

L'Autore, estendendo suoi precedenti risultati [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1957), 188–191, 339–344; MR **16**, 1113], dimostra due teoremi di unicità per il problema di Cauchy, con lo steno tipo di ipotesi introdotte da Caratheodory per le equazioni ordinarie. I teoremi dimostrati concernono il sistema in $m+1$ variabili indipendenti

$$\frac{\partial z_j}{\partial x}=f_j\left(x, y_1, \cdots, y_m, z_1, \cdots, z_r, \frac{\partial z_j}{\partial y_1}, \cdots, \frac{\partial z_j}{\partial y_m}\right), j=1, \cdots, r,$$

e il sistema quasi-lineare in due variabili indipendenti

$$\frac{\partial z_j}{\partial x}=f_j(x, y, z_1, \cdots, z_r)\frac{\partial z_j}{\partial y}+g_j(x, y, z_1, \cdots, z_r), j=1, \cdots, r.$$

Notevoli esempi critici illustrano gli enunciati.
L. Amerio (Milano).

Greco, Donato. Un'osservazione sul problema di Dirichlet. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 73–80 (1957).

Data l'equazione di tipo ellittico (in m variabili indipendenti)

$$(*) \qquad M(u)=\sum a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k}+\sum b_k\frac{\partial u}{\partial x_k}+cu=0 \ (c\leqq 0),$$

l'A. assegna delle condizioni sufficienti perchè il problema di Dirichlet ammetta soluzione nella classe $\Gamma^{(p)}$ di funzioni per cui valgono le formule fondamentali di Green, i valori di frontiera risultando di classe $L^{(p)}$. Si vale per questo, tra l'altro, di una caratterissazione di tale classe indicata recentemente de L. Amerio e di un teorema di G. Fichera. I teoremi dimostrati dall'A. sono i seguenti. (I) Se $v(x)$ è assolutamente continua con le sue derivate prime in $T-FT$, la $v(x)$ e le sue derivate prime e seconde riuscendo di classe $L^{(p)}$ in T ($p>1$), nella classe $\Gamma^{(p)}$ esiste una e una sola soluzione $u(x)$ del problema di Dirichlet relativo alla (*) e alla condizione al contorno $u(x)=v(x)$, $x \in FT$. (II) Se $\zeta(x)$ è di classe $L^{(p)}$ su FT ($p>1$), ed $L(x, y)$ è una funzione di Levi relativa all' operatore M, per la quale riesce $\lambda>(p-1)/p$, esiste in $\Gamma^{(p)}$ una e una sola soluzione del problema di Dirichlet relativo alla (*) e alla condizione al contorno $u(x)=\int_{FT} L(x, y)\zeta(y)d_y\sigma$ ($x \in FT$). In quest'ultimo enunciato λ, con $0<\lambda\leqq 1$, è una costante associata alla funzione di Levi e alla soluzione fondamentale $G(x, y)$, dalle formule

$$L(x, y)-G(x, y)=O(\overline{xy}^{2+\lambda-m}),$$

$$\frac{\partial}{\partial x_i}(L(x, y)-G(x, y))=O(\overline{xy}^{1+\lambda-m}),$$

$$\frac{\partial^2}{\partial x_i \partial x_j}(L(x, y)-G(x, y))=O(\overline{xy}^{\lambda-m}).$$

Il dominio T, di integrazione, si suppone di classe $A^{(3)}$.
L. Amerio (Milano).

Hörmander, L.; et Lions, J. L. Sur la complétion par rapport à une intégrale de Dirichlet. Math. Scand. **4** (1956), 259–270.

Solution complète du problème suivant, important dans l'étude du problème de Dirichlet relatif à l'opérateur Δ^m, posé et résolu dans quelques cas particuliers par Deny et Lions [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 305–370; MR **17**, 646]: soit $\mathscr{D}(\Omega)$ l'espace des fonctions à valeurs complexes, indéfiniment dérivables, à support compact dans un ouvert Ω de R^n; à quelles conditions l'espace $\hat{\mathscr{D}}^m(\Omega)$, complété de $\mathscr{D}(\Omega)$ pour la norme $\|u\|_m=(\sum_{|q|=m}\int_\Omega |D^q u|^2 dx)^{\frac{1}{2}}$, est-il un espace de distributions sur Ω?

On montre qu'il en est toujours ainsi si $n\geqq 2m+1$; si n est impair $<2m$, il faut et il suffit que le complémentaire $\mathbf{C}\Omega$ ne soit pas vide. Le cas le plus délicat est celui où n est pair et $\leqq 2m$; pour l'étudier on introduit une notion d'ensemble polaire d'ordre p (p réel >0), généralisant celle d'ensemble de capacité nulle (à laquelle elle se réduit pour $p=1$); un tel ensemble est vide pour $p>n/2$; alors: Soit $n=2m$; pour que $\hat{\mathscr{D}}^m(\Omega)$ soit un espace de distributions, il faut et il suffit que le complémentaire $\mathbf{C}\Omega$ ne soit pas polaire d'ordre $n/2$. Soit $n=2m-2k$, k entier >0; pour que $\hat{\mathscr{D}}^m(\Omega)$ soit un espace de distributions, il faut et il suffit que $\mathbf{C}\Omega$ ne soit pas polaire d'ordre $n/2$ ou bien ne soit pas contenu dans un hyperplan. L'outil essentiel est la transformation de Fourier. Lorsque $\hat{\mathscr{D}}^m(\Omega)$ est un espace de distributions, ses éléments sont des fonctions dont on signale quelques propriétés de sommabilité ou de continuité. *J. Deny* (Strasbourg).

Malgrange, B. Sur l'intégrale de Dirichlet. Math. Scand. **4** (1956), 271–275.

Généralisant les notations du travail analysé ci-dessus, on appelle $\hat{\mathscr{D}}^m_\Omega(D)$ l'espace obtenu en complétant $\mathscr{D}(\Omega)$ pour la norme $\|f\|_m=(\int D(f, f)dx)^{\frac{1}{2}}$, avec

$$D(f, g)=\sum_{i,j} a_{ij}D^i f \bar{D}^j g,$$

où $|i|=m$, $D^i=\partial^m/\partial x_1^{i_1}\cdots\partial x_n^{i_n}$, les a_{ij} sont des fonctions indéfiniment dérivables dans Ω, enfin la forme hermitienne associée $\sum a_{ij}(x)\xi_i\bar{\xi}_j$ est définie positive pour tout $x\in\Omega$.

L'auteur considère l'ensemble (linéaire) P des polynomes p de degré $\leqq m-1$ possédant la propriété suivante: il existe une suite de fonctions $f_t\in\mathscr{D}(\Omega)$ qui tendent vers 0 dans $\hat{\mathscr{D}}^m_\Omega(D)$ et vers p dans $\mathscr{L}^m_\Omega$ (espace des fonctions de carré sommable ainsi que toutes leurs dérivées d'ordre $\leqq m$ sur tout compact de Ω). Le problème fondamental relatif à $\hat{\mathscr{D}}^m_\Omega(D)$ est la détermination de P (par exemple si les a_{ij} sont constants et $\Omega=R^n$, on a $P=\{0\}$ si $n>2m$ et P=ensemble des polynomes de degré $\leqq m-[\frac{1}{2}(n+1)]$ si $n\leqq 2m$). Sans aborder l'étude de ce difficile problème dans le cas général, l'auteur donne des conditions nécessaires et suffisantes pour que l'opérateur D admette un noyau de Green dans $\Omega-K$, où K est un compact de Ω. *J. Deny.*

Huet, Denise. Phénomènes de perturbation singulière. C. R. Acad. Sci. Paris **244** (1957), 1438–1440.

Soient A et B deux opérateurs différentiels sur un ouvert Ω de R^n; A [B] est défini par une forme sesquilinéaire continue $a(u, v)$ [$b(u, v)$] sur un espace de Hilbert V [W] contenu dans $L^2(\Omega)$, ce qui définit A et B et des problèmes aux limites correspondants. On suppose ces formes V et W elliptiques ($\mathrm{Re}\, a(v, v)\geqq\alpha\|v\|^2$, $\alpha>0$, id. pour b). L'opérateur A est d'ordre plus grand que B (en termes abstraits: $V\subset W$). Soit u_ε la solution de $(\varepsilon A+B)u_\varepsilon=f$, avec certaines conditions aux limites définies par V et les formes a, b. Si V est dense dans W, $u_\varepsilon\to u$ dans W lorsque $\varepsilon\to 0$, u étant la solution de $Bu=f$ avec les conditions aux limites définies par b et W. Exemples.

Résultats analogues pour les opérateurs $\varepsilon A+B+\partial/\partial t$ ou $\varepsilon A+B+\partial^2/\partial t^2$, et les problèmes mixtes associés.

J. L. Lions (Nancy).

Cattabriga, Lamberto. Metodo di separazione delle variabili e problema generalizzato di Dirichlet. Boll. Un. Mat. Ital. (3) **11** (1956), 49–54.

The main theorem of this paper establishes the existence of a function $u(P)$ which is harmonic in the interior of a bicylindrical domain $D\subset R^{n+m}$ and, in a certain sense, assumes given values f "in the mean" on the boundary of D; f is a given function, square summable on the boundary, FD, of D. Instead of considering Cartesian coordinates for the points $(x_1, \cdots, x_n, y_1, \cdots, y_m)$ of D, where $\sum_{i=1}^n x_k^2\leqq a^2$ and $\sum_{j=1}^m y_j^2\leqq b^2$, spherical coordinates $(\rho, \varphi_1, \cdots, \varphi_{n-1}; \sigma, \psi_1, \cdots, \psi_{m-1})$ where $\rho\leqq a$; $\sigma\leqq b$ are used. FD is then the union of 2 sets S_1: $\rho=a$, $\sigma\leqq b$, and S_2: $\rho\leqq a$, $\sigma=b$. Then D^k is defined to be the subdomain whose boundary is the union of the sets S_1^k: $\rho=a-k$, $\sigma\leqq b-k$, and S_2^k: $\rho\leqq a-k$, $\sigma=b-k$. To say that $u(P)$ "assumes the values of f in the mean on FD" means that

$$\lim_{k\to 0+}\int FD^k[u(Q^{(k)})-f(Q)]^2 d\omega^{(k)}=0.$$

The method of proof consists in first establishing uniqueness, and then, by means of Bessel functions and ultraspherical polynomials, constructing systems of functions which are complete and orthogonal on S_1 and S_2, and finally constructing a sequence of functions which converges uniformly to a harmonic function $u(P)$ in $D-FD$ and has the required property on FD. *D. L. Bernstein.*

★ **Cimmino, Gianfranco. Una estensione dei teoremi di convergenza e di unicità nella teoria del problema generalizzato di Dirichlet.** Scritti matematici in onore di Filippo Sibirani, pp. 87–94. Cesare Zuffi, Bologna, 1957.

Sia D un dominio piano limitato da una curva semplice e chiusa di ascissa curvilinea s e lunghezza l. Sia poi D_t un dominio interno a D approssimante D per $t\to 0$ e siano: $x=x(s, t)$, $y=y(s, t)$, le equazioni parametriche della frontiera di D_t. Il risultato principale di questo lavoro è il seguente: Se una funzione u, armonica in $D-\mathscr{F}D$, è tale che per ogni $g(s)\in\mathscr{L}^2$ riesce:

$$(*)\qquad \lim_{t\to 0}\int_0^l u(x(s, t), y(s, t))g(s)ds=0$$

si ha $u\equiv 0$ in D. L'A. indica anche varie possibilità di estendere questo risultato sia sostituendo alla (*) un'ipotesi meno restrittiva che considerando un'equazione più generale di quella delle funzioni armoniche.

C. Miranda (Napoli).

★ **Miranda, Carlo. Equazioni alle derivate parziali di tipo ellittico.** Ergebnisse der Mathematik und ihrer Grenzgebiete (N. F.), Heft 2. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+222 pp. DM .28.80; $6.85.

Questa Monografia costituisce, essenzialmente, una rassegna completa ed approfondita dei vari metodi che sono stati introdotti nella letteratura matematica per conseguire i teoremi di esistenza nei problemi riguardanti le equazioni alle derivate parziali del secondo ordine di tipo ellittico lineari e non lineari. Con potenza di sintesi e perspicuità di esposizione veramente mirabili, L'A. riesce in tale intento, fornendo ai cultori di equazioni alle derivate parziali un'opera da ritenersi di fondamentale importanza. Dopo un primo capitolo dedicato alle nozioni più semplici sulle equazioni lineari ellittiche (teoremi di unicità, formole di reciprocità, funzioni di Levi, di Green, fondamentali etc.), il secondo capitole, di per sè interessante, comprende i più approfonditi e completi risultati di una teoria del potenziale, concepita da un punto di vista generale e assai moderno. Ciò prepara il campo agli argomenti svolti nel terzo capitolo, che contiene l'analisi esistenziale per i vari problemi al contorno, relativi alle equazioni ellittiche lineari, conseguita con il metodo dei potenziali e delle equazioni integrali secondo l'indirizzo sviluppato principalmente da Giraud. Chi ha avuto agio di studiare le numerose e ponderose Memorie di questo Autore, peraltro pregevolissime, non può non restare ammirato della limpida esegesi che di esse fa l'A. nella Sua trattazione, che costituisce un vero e proprio contributo all'argomento. Segnaliamo, a mo' d'esempio, il teorema di esistenza per la soluzione fondamentale principale e l'indagine relativa ai problemi di derivata obliqua. L'A. intitola il successivo capitolo „Soluzioni generalizzate dei problemi al contorno". In esso trova largo posto un metodo fondato su una idea di Caccioppoli e sviluppata da Cimmino e da altri, che trae partito da un'opportuna generalizzazione del concetto di „condizione al contorno", metodo esposto dall'A. con ricchezza di particolari e di risultati. Vengono anche considerati il metodo cosidetto delle proiezioni ortogonali, quello dovuto al Picone, che l'A. chiama delle equazioni di Fischer-Riesz, e quello variazionale di Ritz. I primi due hanno dato luogo a tante ricerche nella più recente letteratura. In effetti il metodo di Picone, a differenza di tutti gli altri metodi di calcolo della soluzione, consente, fra l'altro, non solo il calcolo di essa e delle sue derivate nei punti interni di un campo, ma anche sulla frontiera di esso, risolvendo per la prima volta un problema posto addirittura dalla tecnica, quale ad esempio quello del calcolo delle reazioni vincolari sulle parti incastrate di un sistema elastico. Inoltre i teoremi di unicità dei problemi generalizzati cui il metodo

di Picone dà luogo, sarebbero di per sè privi di interesse, ma essi costituiscono la dimostrazione di altrettanti teoremi di completezza, esplicitamente formulabili, che sono alla base delle più intime connessioni essistenti fra i problemi al contorno e l'analisi funzionale lineare. Il capitolo V è dedicato alle formole di maggiorazione „a priori" per le soluzione delle equazioni ellittiche. È questo uno dei più suggestivi ed essenziali argomenti della teoria e trae partito dalle tecniche più raffinate impiegate nello studio delle equazioni ellittiche. L'esposizione dei risultati, principalmente dovuti a Schauder e Caccioppoli, relativi alla maggiorazione del modulo e dei coefficienti di Hölder delle soluzioni e delle derivate di esse, è fatta con il massimo di rigore e compiutezza. Una prima applicazione di tali risultati è costituita dal metodo del prolungamento funzionale per la dimostrazione del teorema d'esistenza relativo al problema di Dirichlet per le equazioni lineari, dovuto a Schauder e Caccioppoli. Successivamente vengono considerate le formole di maggiorazione integrale per le soluzioni delle equazioni ellittiche, che tanti notevoli contributi hanno permesso di apportare alle teoria in questi ultimi anni, e, infine, viene considerato il caso di equazioni con coefficienti soltanto continui o con termine noto di quadrato sommabile, relativamente alle quali si devono proprio all'A. pregevoli risultati. Il capitolo VI è dedicato principalmente allo studio del problema di Dirichlet per le equazioni non lineari. Il difficile e delicatissimo argomento è trattato con vera maestria ed è certo un grande titolo di merito dell'A. l'aver saputo rielaborare in una piana esposizione la materia di fondamentali Memorie sull'argomento che, talvolta, forse per la loro stringatezza, erano state criticate a torto da qualche autore. La trattazione dei teoremi d'esistenza è fatta in modo da esser accessibile a tutti ed è, per ciò, opportunamente preceduta da utili richiami sui metodi esistenziali fondati sulla topologia funzionale. Viene infine considerato il problema di Hilbert sulle soluzioni analitiche ed esposti i principali risultati al riguardo. L'ultimo capitolo è più che altro una rassegna bibliografica di diversi argomenti alle equazioni ellittiche; equazioni ellittiche su una varietà, problemi in domini non limitati, problemi misti, sistemi ellittici, equazioni di ordine superiore etc. È augurabile che in una eventuale prossima edizione dell'Opera, qualcuno di questi argomenti trovi più cospicui sviluppi. Ad esempio lo studio del problema misto di Dirichlet-Neumann, qui solo limitato a semplici indicazioni bibliografiche, e quello dei sistemi lineari ellittici del primo ordine. Tanto più che proprio questo ultimo argomento deve all'A. notevolissimi risultati. Chiude la Monografia una imponente raccolta bibliografica.

G. Fichera (Zbl. 65 (1956), 85).

Foiaş, Ciprian; Gussi, George; et Poenaru, Valentin. Sur le problème de Cauchy pour le type elliptique à deux variables. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. 7 (1955), 97–103. (Romanian. Russian and French summaries)

The authors use Hadamard's example, $u_{xx}+u_{yy}=0$, $u(x, 0)=0$, $u_y(x, 0)=n^{-k}\sin nx$, $k\geqq 1$, and also analogous examples for other equations, to show that, if attention is restricted to the real domain, the Cauchy problem is not properly set for elliptic partial differential equations, in that one does not get stability: the solutions do not depend in a continuous way on the initial data. This much is not new [cf. I. G. Petrovsky, Lectures on partial differential equations, Gostehizdat, Moscow, 1950, pp. 80 ff.; MR 13, 241; 16, 478; see also p. 166 of the book reviewed above]. The novelty of the present paper, and its clarification of this important problem, lies in what follows. Working with the analytic continuations of the initial data into the complex domain, the authors prove existence and uniqueness for the solution of the Cauchy problem for certain elliptic equations, and also prove stability (the real contribution lies precisely here); i.e., the solution is continuous in the initial data, if the analytic continuations of the initial data are used in the criterion for convergence. In examples of Hadamard's type, these analytic continuations of the initial data do not converge (e.g., $n^{-k}\sin nx$ for complex x). (The situation suggests the familiar relation between singularities of functions and radius of convergence of power series.) The proof of existence depends upon solving a non-linear integral equation by a method of successive approximations.

R. B. Davis (Durham, N.H.).

★ **Schwartz, Laurent. Ecuaciones diferenciales parciales elipticas. [Elliptic partial differential equations.]** Universidad Nacional de Colombia, Departamento de Matemáticas y Estadística, Bogotá, D. E., Colombia. 82 pp. (mimeographed) $1.00.

Un opérateur différentiel

$$(1)\quad D=\sum_{|p|\leq m}\alpha_p(x)D^p \quad (x\in\mathscr{R}^n,\ D^p=D_{x_1}^{p_1}\cdots D_{x_n}^{p_n}),$$

où les $\alpha_p(x)$ sont indéfiniment dérivables, est dit elliptique, si, pour tout x, l'expression $\sum\alpha_p(x)\zeta^p$, avec $|p|=m$, est une forme définie en $\zeta\in\mathscr{R}^n$. On dit que D est hypoelliptique, si toute distribution T, telle que DT est une fonction indéfiniment dérivable, est aussi une fonction indéf. dérivable. Ces notions se généralisent au cas où les $\alpha_p(x)$ sont à valeurs dans un espace de matrices finies: $DX=A$ est alors un système d'équations scalaires.

Ce cours a pour objet central le théorème suivant: tout opérateur elliptique est hypoelliptique. Dans le § 2 on introduit plusieurs notions et résultats conduisant à ce théorème. Pour chaque entier $s>0$, $\mathfrak{H}^s$ est l'espace hilbertien des distributions f telles que $D^pf\in L^2(\mathscr{R}^n)$, pour $0\leq|p|\leq s$; le produit scalaire sur $\mathfrak{H}^s$ est défini par $((f, g))_s=\sum\int f^{(p)}(x)\bar{g}^{(p)}(x)dx$, sur $\mathscr{R}^n$. Cette définition se généralise au cas d'un nombre s réel quelconque, moyennant la transformation de Fourier. Le dual de $\mathfrak{H}^s$, réalisé comme "espace de distributions", s'identifie à $\mathfrak{H}^{-s}$, constitué par les distributions de la forme $\sum D^pf_p$, $|p|\leq s$, $f_p\in\mathscr{L}^2$, lorsque s est entier >0. (Il y a ici un "tournant dangereux", tenant à la définition fonctionnelle de distribution: le dual de $\mathfrak{H}^s$ n'est pas à confondre ici avec le dual de l'espace hilbertien $\mathfrak{H}^s$ au sens usuel.) Ces espaces, et d'autres que l'on en déduit d'une façon simple, interviennent dans plusieurs théorèmes qui impliquent le th. fondamental, démontré au § 3.

Dans le § 4 l'auteur rappelle quelques notions fondamentales sur les espaces fibrés de fibre type $\mathscr{C}^k$ et il introduit la notion de section distribution, généralisant celles de distribution et de courant. Dans le § 5 il définit "opérateur différentiel" entre deux espaces fibrés V_1, V_2, de fibres vectorielles F_1, F_2, comme application linéaire continue D de $\mathfrak{E}(V_1)$ (espace des sections globales indéf. dérivables de V_1) dans $\mathfrak{E}(V_2)$, telle que le support de Df est toujours contenu dans le support de f. Sur chaque domaine de carte locale, D reprend la forme concrète (1), où les $\alpha_p(x)$ sont à valeurs dans $L(F_1, F_2)$. L'auteur donne également une définition intrinsèque de "opérateur elliptique", qui sur toute carte locale reprend sa forme initiale. Comme exemple, on étudie (§ 6) l'opérateur de

Laplace sur une variété de Riemann orientée. Le th. fondamental se généralise immédiatement aux espaces fibrés. Dans le § 7 on établit quelques propriétés plus fines des opérateurs elliptiques sur ces espaces.

Le § 8 est consacré à l'étude des opérateurs A-elliptiques, c.à.d., elliptiques et tels que toute solution de $Df=0$, dans un ouvert connexe Ω quelconque, s'annulant dans un ouvert $\omega \subset \Omega$, est partout nulle dans Ω [cf. Malgrange, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355; MR **19**, 280]. Enfin, dans le § 9 l'auteur pose quelques problèmes non résolus; par exemple: tout opérateur elliptique est-il A-elliptique?

Ce cours a été accompagné, à Bogota, par un cours de Mme M.-H. Schwartz sur les espaces fibrés et par un autre de L. Schwartz sur les variétés analytiques complexes. *J. Sebastião e Silva* (Lisbonne).

Karp, Samuel N. An application of Sturm-Liouville theory to a class of two-part boundary-value problems. Proc. Cambridge Philos. Soc. **53** (1957), 368–381.

The problem of propagation in a symmetrically bifurcated parallel plate wave guide symmetrically filled with a material of variable dielectric constant is considered. This reduces to the boundary value problem

$$\Delta u + k^2(y)u = 0 \quad (0<y<a,\ -\infty<x<\infty) \tag{1}$$

with

$$u(x,0)=0 \quad (-\infty<x<\infty), \tag{2}$$
$$u_y(x,a)=0 \quad (x>0), \qquad u(x,a)=0 \quad (x<0).$$

The function $k(y)$ is symmetric about $y=a$. It is assumed that there is a single propagating mode for $x>0$, and none for $x<0$. This means that the eigenvalue problem

$$v(y)''+(\lambda+k^2)v=0,\ v(0)=v'(a)=0 \tag{3}$$

has exactly one negative eigenvalue λ_0 and positive eigenvalues $\lambda_1<\lambda_2<\cdots$, while the problem

$$w(y)''+(\mu+k^2)w=0,\ w(0)=w(a)=0 \tag{4}$$

has only positive eigenvalues $\mu_1<\mu_2<\cdots$.

Letting $\varphi(-\nu^2, y)$ be the solution of

$$\varphi''+(-\nu^2+k^2)\varphi=0,\ \varphi(0)=0,\ \varphi'(0)=1, \tag{5}$$

the author combines the two expansions

$$u(x,y)=\begin{cases} (c_0^{(1)}e^{ix(-\lambda_0)^{1/2}}+d_0^{(1)}e^{-ix(-\lambda_0)^{1/2}})\ \varphi(\lambda_0, y)+\sum_{n=1}^{\infty} c_n^{(1)}e^{-x\lambda_n^{1/2}}\varphi(\lambda_n, y) & \text{for } x>0, \\ \sum_{n=1}^{\infty} c_n^{(2)}e^{x\mu_n^{1/2}}\varphi(\mu_n, y) & \text{for } x<0, \end{cases} \tag{6}$$

into a single Fourier transform

$$u(x,y)=[2\pi i]^{-1}\int_C e^{i\nu x}\varphi(-\nu^2, y)A(\nu)\,d\nu, \tag{7}$$

where C lies along the real axis, but passes below the points $\pm(-\lambda_0)^{1/2}$. In order to match the expansions (6), $A(\nu)$ must have poles at $\nu=\pm(-\lambda_0)^{1/2}$, $i\lambda_n^{1/2}$, and $-i\mu_n^{1/2}$ $(n=1, 2, \cdots)$. Since $u(x,a)\sim x^{1/2}$ as $x\to+0$, $A(\nu)=O(\nu^{3/2})$ as $\nu\to\infty$. From these facts, the author constructs the function $A(\nu)$ in terms of infinite products. $A(\nu)$ is thus exhibited as a product of functions having poles only in the upper and lower half-plane, respectively. Hence the method is related to, but more direct than, the Wiener-Hopf method.

The author states without details that the method can be used for more general, including non-symmetric, situations. *H. F. Weinberger* (Madison, Wis.).

Babakova, O. I. Generalized problem of torsion for a two-dimensional multiangular region. Ukrain. Mat. Ž. **8** (1956), 450–453. (Russian)

Given simply or multiply-connected region D with boundary consisting of arcs L_i. Also given polynomials $P_i(x, y)$ each of which corresponds to an arc and where appropriate values of the corresponding polynomials coincide if two arcs have a common point. To find the analytic function $F(z)$, continuous to the boundary, which is regular in D and for which the imaginary part $\psi(x, y)$ satisfies the boundary condition $\psi(x, y)=P_\nu(x, y)+C$ on the νth arc, where C is a different constant in each connected part of the boundary. Problem was solved by P. P. Kufarev [Prikl. Mat. Meh. (N.S.) **1** (1937), 43–76] for simply-connected multiangular region. Author solves it for doubly-connected region by mapping doubly-connected region D into an annulus by using an elliptic function formula of N. I. Akhiezer. Analytic continuation of solution beyond the annulus leads to an elliptic function representable by θ-functions. Solution obtained is used to solve torsion problems. *M. D. Friedman.*

Kozyrev, I. I. Solution of Dirichlet problems for doubly connected polygonal domains. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 35–39. (Russian)

Author uses another method to solve problem solved by Babakova in article reviewed above. *M. D. Friedman.*

Sbrana, Francesco. Sulla integrazione di certe equazioni differenziali lineari alle derivate parziali. Rend. Sem. Mat. Fis. Milano **26** (1954–55), 116–121 (1957).

The author discusses the solution of two partial differential equations by means of functional transformations allied to the Laplace transform. In the non-homogeneous wave equation

$$\nabla^2 U=\frac{1}{V^2}\frac{\partial^2 U}{\partial t^2}-\Phi(x, y, z, t),$$

where Φ is a given function, the transformation is

$$u(x, y, z, t;\ \omega)=\iiint_V U(\xi, \eta, \zeta, t)\,\frac{e^{\omega R}}{R}\,d\xi d\eta d\zeta \quad (\omega=c+i\lambda),$$

where (x, y, z) is a point of the volume V in which the equation holds and $R^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$.

For the equation

$$\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}=\frac{\partial^2\Phi}{\partial y_1^2}+\frac{\partial^2\Phi}{\partial y_2^2} \quad (-a\le x_i\le a;\ -b\le y_i\le b),$$

the transformation is

$$\varphi(x_1, x_2, y_1, y_2;\ r, \rho)=\int \Phi(\xi_1, \xi_2, \eta_1, \eta_2)J_0(rX)J_0(\rho Y)\,d\xi_1 d\xi_2 d\eta_1 d\eta_2,$$

where integration is over $-a\le\xi_i\le a$, $-b\le\eta_i\le b$, and $X^2=(x_1-\xi_1)^2+(x_2-\xi_2)^2$, $Y^2=(y_1-\eta_1)^2+(y_2-\eta_2)^2$.

The paper is really only a very brief report of a lecture, and more detail is needed before one can appreciate the importance of the ideas. The author's notation has been slightly changed. *E. T. Copson* (St. Andrews).

Yosida, Kôsaku. An operator-theoretical integration of the temporally inhomogeneous wave equation. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 463–466.

In this paper a new method is described for solving the Cauchy problem for the wave equation $u_{tt}-Au=0$, $u(0)=f$, $u_t(0)=g$, where $A(t)$ is an elliptic operator of

second order whose coefficients depend on x and t. Introducing $v=u_t$ as another unknown we can write the equation as $U_t=B(t)U$, $U(0)=F$, where U denotes the column vector (u, v), B the matrix $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$. The norm of U is taken as the kth order Dirichlet integral of u plus the $(k-1)$st order Dirichlet integral of v. From the theory of elliptic equations it follows, after imposing appropriate boundary conditions on u, that the resolvent $(1-B/n)^{-1}$ exists for $|n|$ sufficiently large. Its norm is $1+O(1/n)$ and it is a strongly continuous function of t. The approximate equation $U_t^{(n)}=B^{(n)}(t)U^{(n)}$, with $B^{(n)}=B(1-B/n)^{-1}$, is set up. Since $B^{(n)}$ is bounded, the initial value problem for it can be solved by iteration. From the estimate for the resolvent of B it follows that $\|U^{(n)}(t)\| \leqq \exp O(t)\|U^{(n)}(0)\|$. $U(t)$ is then constructed as the weak limit of $U^{(n)}$. Its differentiability is demonstrated with the aid of Soboleff's lemma. *P. D. Lax* (New York, N.Y.).

Yosida, Kôsaku. An operator-theoretical integration of the wave equation. J. Math. Soc. Japan **8** (1956), 79–92.

In dieser Abhandlung gibt Verf. eine weitere Anwendung seiner Theorie der 1-gliedrigen Gruppen linearer Operatoren, und zwar löst er das Cauchy-Problem im ganzen m-dimensionalen Raume E^m für die folgende hyperbolische Gleichung: (1) $\partial^2 u(x, t)/\partial t^2 = Au(x, t)$, $u(x, 0)=f(x)$, $\partial u(x, 0)/\partial t = g(x)$, wobei

$$Au \overset{\mathrm{df}}{=} \sum_{i,k=1}^{m} a^{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u;\ x \in E^m.$$

Es soll hervorgehoben werden daß das Problem ohne Benutzung des Cauchy-Kowalewski-Satzes gelöst wurde. Hauptergebnis ist wie folgt. Voraussetzungen: 1. Die Koeffizienten von A sind reell und beliebig oft differenzierbar; 2. $\max(\sup|a^{ij}(x)|, \sup|b^i(x)|, \sup|c(x)|, \sup|\partial a^{ij}/\partial x_k|, \sup|\partial b^i/\partial x_k|, \sup|\partial^2 a^{ij}/\partial x_i \partial x_j|) < \infty$; 3. Es gibt solche positive Konstanten μ_1, μ_2, daß $\mu_1 \sum \xi_i^2 \leqq \sum a^{ij}(x)\xi_i\xi_j \leqq \mu_2 \sum \xi_i^2$; 4. $f, g \in C^\infty$, $A^k f$, $A^k g$, $\partial A^k g/\partial x_i \in L^2(E^m)$, $k=1, 2, \cdots$. Behauptung: Es gibt eine solche positive Konstante β, daß für hinreichend kleines $\alpha_0>0$ das Cauchyproblem (1) eine einzige Lösung $u \in C^\infty$ besitzt, wobei u der Ungleichung

$$[(u-\alpha_0 Au, u)+\alpha_0(\partial u/\partial t, \partial u/\partial t)]^{\frac{1}{2}} \leqq \exp \beta|t|[(f-\alpha_0 Af, f)+\alpha_0(g, g)]^{\frac{1}{2}}$$

genügt, wo $(h, g) \overset{\mathrm{df}}{=} \int_{E^m} h(x)g(x)dx$. Als interessantes Zwischenprodukt bekommt der Verf. natürliche Ungleichungen und das folgende Lemma: Für hinreichend kleines $\alpha_0>0$ besitzt die Gleichung $u-\alpha Au=f$, $0<\alpha \leqq \alpha_0$, für jedes $f \in H$ eine einzige Lösung $u \in H$, wo

$$H \overset{\mathrm{df}}{=} \left\{ f \in C^\infty : \int_{E^m} f^2 dx + \int_{E^m} \sum \left|\frac{\partial f}{\partial x_i}\right|^2 dx < \infty \right\}.$$

K. Maurin (Zbl. **71** (1957), 92).

Lions, J. L. Une remarque sur les applications du théorème de Hille-Yosida. J. Math. Soc. Japan **9** (1957), 62–70.

The author discusses the mixed problem, on an open domain Ω of an n-dimensional euclidean x-space, for the operator $A+B\partial/\partial t+C\partial^2/\partial t^2$, where A is an elliptic operator in x of order $2m$, B an operator in x of order m and C an invertible operator in x of order 0. Let H^m be the totality of $L_2(\Omega)$-functions u such that the distribution derivatives $D^p u \in L_2(\Omega)$ for $D^p = \partial^{p_1+\cdots+p_m}/\partial x_1^{p_1} \cdots \partial x_m^{p_m}$ and $|p| = \sum_i p_i \leqq m$; H^m is thus a Hilbert space with the norm $\|u\|_m = (\sum_{k=0}^m |u|_k^2)^{\frac{1}{2}}$, where $|u|_k^2 = \sum_{|p|=k} |D^p u|_0^2$, $|u|_0^2 = (u, \bar{u})_0 = \int_\Omega |u(x)|^2 dx$. Let H_0^m be the strong closure in H^m of the space $D(\Omega)$ of the totality of C^∞ functions on Ω with compact supports. Ω is supposed to be m-regular in the sense that there exist positive constants a and b satisfying $|u|_k \leqq a|u|_m + b|u|_0$ for $1 \leqq k \leqq m-1$, $u \in H^m$. Let V be a closed subspace of H^m such that $H_0^m \subset V \subset H^m$, and let, for any $u \in V$, the value $\langle Au, \bar{v}\rangle = a(u, v)$ of the distribution Au at the testing function $\bar{v} \in D(\Omega)$ satisfy the condition (H1): the Hermitian part $a_0(u, v)$ of the sesquilinear form $a(u, v)$ on $V \times V$ satisfies $a_0(u, v) \geqq c_0|u|_m^2$ and $|\mathrm{Re}(a(u, v)-a_0(u, v))| \leqq c_1\|u\|_m|v|_0$. It is further assumed that B and C, considered respectively as operators from V into $H^0=L_2(\Omega)$ and from H^0 into H^0, satisfy (H2) $|\mathrm{Re}(Bv, v)_0| \leqq c_3|v|_0^2$ for $v \in V$, and (H3) $(Cf, f)_0 \geqq c_4|f|_0^2$ for $f \in H^0$. Let $N \supseteq D(\Omega)$ be the totality of $u \in V$ such that $Au \in H^0$ and $(Au, v)_0 = a(u, v)$ for every $v \in V$. Then it is proved that the mapping $\{u, v\} \to \mathfrak{A}\{u, v\} = \{v, -C^{-1}Au - C^{-1}Bv\}$ from $N \times V$ into $V \times H^0$ is such that, for sufficiently large $|\lambda|$, λ real, $(I-\lambda^{-1}\mathfrak{A})$ is an isomorphism of $N \times V$ onto $V \times H^0$. Moreover, the inverse $G_\lambda = (I-\lambda^{-1}\mathfrak{A})^{-1}$, considered as an operator from $V \times H^0$ into $V \times H^0$, satisfies $|G_\lambda| \leqq 1+|\lambda^{-1}|\beta$, $\beta>0$. Here the norm in $V \times H^0$ is defined by $|\{u, v\}|^2 = a_0(u, u)+(Cu, u)_0+(Cv, v)_0$. Thus $\mathfrak{A}$ is the infinitesimal generator of a group $X(t)$, $-\infty<t<\infty$, of bounded linear operators on $V \times H^0$ into $V \times H^0$, so that, for $f \in N$, $g \in V$, $X(t)\{f, g\} = \{u(t), v(t)\}$ satisfies $du(t)/dt = v(t)$, $dv(t)/dt = -C^{-1}(Au(t)+Bv(t))$ with the initial condition $u(0)=f$, $v(0)=g$. Thus the result extends those of Lax [Bull. Amer. Math. Soc. **58** (1952), 182] and the reviewer [see the paper reviewed above]. *K. Yosida* (Tokyo).

Ciliberto, Carlo. Su alcuni problemi relativi ad una equazione di tipo iperbolico in due variabili. Boll. Un. Mat. Ital. (3) **11** (1956), 383–393.

In two previous papers [Richerche Mat. **4** (1955), 15–29; Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 221–225; MR **17**, 621; **19**, 149] the author has given proofs of the existence of a solution of the Darboux problem for the equation: (*) $z_{xy}=f(x, y, z, z_x, z_y)$. Using similar methods, in the present paper, he establishes the existence of solutions of equation (*) which satisfy two other sets of boundary conditions. In one problem, the value of z is given along a curve C_1 and the value of z_x along a curve C_2; in the other problem, the value of z_x is given along C_1, the value of z_y along C_2, and the value of z at one point. Exact hypotheses are too long to state here. As in the preceding papers, there are preliminary theorems concerning the equicontinuity of a set of functions which are all solutions of a certain ordinary differential equation for various parameter functions. *D. L. Bernstein.*

Hornich, H. Zur Lösbarkeit der hyperbolischen Differentialgleichungen. Österreich. Ing.-Arch. **10** (1956), 195–197.

Using results previously established by the author on the nonsolvability of first-order partial differential equations [Monatsh. Math. **59** (1955), 34–42; Rend. Sem. Mat. Univ. Padova **24** (1955), Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 33–37; MR **16**, 825; **17**, 489, 620], the following theorem is proved. To every hyperbolic linear second order equation, there corresponds another equation, whose coefficients are arbitrarily close to those of the given equation, but which is non-solvable in a given region of the (x, y) plane. *D. L. Bernstein.*

Fulks, W. Regular regions for the heat equation. Pacific J. Math. 7 (1957), 867–877.

An open connected set R in the plane or in space is said to be a regular region for Laplace's equation $\Delta u=0$ if the Dirichlet problem for R has a unique solution for continuous data. R is said to be a regular region for the heat equation $\Delta u=u_t$ if the "Dirichlet problem" for it has a unique solution for continuous data (viz., u given on the closure of R for $t=0$, u given on the boundary of R for $t\geqq 0$). A. Tychonoff [Bull. Univ. d'Etat Moscou. Sér. Internat., Sect. A. 1 (1938), no. 9] proved that if R is bounded and regular for Laplace's equation, it is regular for the heat equation; and conversely. A new proof of this result is given here. *E. T. Copson* (St. Andrews).

Pogorzelski, W. Propriétés des intégrales de l'équation parabolique générale. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 407–410.

In an earlier paper [same Bull. 4 (1956), 9–13; Ricerche Mat. 5 (1956), 25–57; MR 18, 47] the author proved the existence of a fundamental solution $\Gamma(x, t; \xi, \tau)$ of the parabolic equation

$$(1)\quad \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x,t)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}+\sum_{\alpha=1}^{n} b_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}+c(x,t)u-\frac{\partial u}{\partial t}=0$$

under rather weak continuity conditions on the coefficients of the equation (1). Here the point $x=(x_1, \cdots, x_n)$ lies in a bounded domain Ω whose bounding surface S satisfies the conditions of Liapounov; and t lies in a fixed range $0\leqq t\leqq T$. In the present paper the author announces properties of the following solutions of (1):

$$(2)\quad U(x,t)=\int_0^t\int\int_S \Gamma(x,t;\xi,\tau)\varphi(\xi,\tau)d\sigma_\xi d\tau,$$

$$(3)\quad J(x,t,\tau)=\int\int\int_\Omega \Gamma(x,t;\xi,\tau)g(\xi,\tau)dv_\xi \quad (0<\tau<t\leqq T),$$

$$(4)\quad V(x,t)=\int_0^t J(x,t,\tau)d\tau.$$

Let P be a point on the surface S and define the transversal derivative of $U=U(x, t)$ as follows:

$$\frac{dU}{dT_P}=\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x,t)\cos(N_P,x_\beta)U_{x_\alpha}.$$

Several of the stated results are given here. (i) If the function $\varphi(\xi, \tau)$ in (2) is continuous, then

$$\lim_{x\to P}\frac{dU(x,t)}{dT_P}=-f(P,t)\varphi(P,t)+\int_0^t\int\int_S \frac{d\Gamma(P,t;\xi,\tau)}{dT_P}\varphi(\xi,\tau)d\sigma_\xi d\tau.$$

Here $f(P, t)$ depends only on the coefficients $a_{ij}(P, t)$ in (1). (ii) If $\varphi(\xi, \tau)$ and $g(\xi, \tau)$ are bounded and integrable functions, then $U(x, t)$ in (2) and $V(x, t)$ in (4) satisfy Hölder conditions. For the function $U(x, t)$ the Hölder condition holds on the closed set $\Omega+S$, $0\leqq t\leqq T$. (iii) If $g(\xi, \tau)$ is continuous, then $\lim_{\tau\to t} J(x, t, \tau)=2f(x, t)g(x, t)$. Proofs of these results are to appear in the Annales Polonici Mathematici. *F. G. Dressel* (Durham, N.C.).

Mambriani, Antonio. Sul concetto di "specie" di un'equazione differenziale. Riv. Mat. Univ. Parma 7 (1956), 141–144.

In un precedente lavoro [la stessa Riv. 6 (1955), 321–348; MR 19, 37] l'A. ha proposto una classificazione delle equazioni differenziali in specie diverse, in relazione col numero dei pluriderivatori necessari per scrivere l'equazione. Tale classificazione viene ora lievemente modificata.

C. Miranda (Napoli).

★ **Mambriani, Antonio. Un saggio di risoluzione, con la pluriderivazione, di certe equazioni differenziali del secondo ordine non lineari.** Scritti matematici in onore di Filippo Sibirani, pp. 161–165. Cesare Zuffi, Bologna, 1957.

In un precedente lavoro [Riv. Mat. Univ. Parma 6 (1955), 321–348; MR 19, 37] l'A. ha introdotto certi operatori differenziali lineari da lui detti pluriderivatori, mostrando come si possa trarre profitto di certe loro proprietà formali, analoghe a quelle dell'operatore derivata, per risolvere alcuni tipi di equazioni differenziali. La presente nota contiene nuovi esempi di tale procedimento.

C. Miranda (Napoli).

Friedman, Avner. On n-metacaloric functions. Proc. Amer. Math. Soc. 8 (1957), 770–776.

Let $u(x, t)$ be a function of $x=(x_1, x_2, \cdots, x_N)$ and t, and let Ω be the operator $\Delta-\partial/\partial t$, where $\Delta=\sum \partial^2/\partial x_i^2$. Let $p(t)$ be a polynomial of degree n. The function $u(x, t)$ is by definition n-metacaloric if $p(\Omega)u(x, t)=0$. Properties are developed which the author describes as analogous to properties of solutions of $p(\Delta)u(x)=0$ studied by Ghermanesco [Bull. Soc. Math. France 62 (1934), 245–264] and by himself [Proc. Amer. Math. Soc. 8 (1957), 223–229; MR 19, 39]. The theory is developed for the case in which $N=1$; it is stated that the results are general. The first two theorems assert: (1) If $p(z)$ is the polynomial $\prod_{i=1}^{k}(z-\alpha_i)^{n_i}$ then in a domain D the general solution of $p(\Omega)u(x, t)=0$ has the form

$$u(x,t)=\sum_{i=1}^{k}\sum_{j=0}^{n_i-1} t^j u_j^{\alpha_i}(x,t),$$

where $u_j^{\alpha_i}(x, t)$ are solutions in D of $(\Omega-\alpha_i)u(x, t)=0$. (2) If, in the strip $t_1\leqq t\leqq t_2$, $(\Omega-\alpha)u(x, t)=0$ and $|u(x, t)|\leqq M\exp Kx^2$, then (for every h, t which satisfy $t_1\leqq t\leqq t_2$, $t_1<t-h$, $0<h<1/8K$) the following mean-value theorem holds:

$$\mu(u,h)=\frac{1}{2\sqrt{(\pi h)}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h}u(\xi,t-h)d\xi=e^{\alpha h}u(x,t).$$

Several further theorems deal with properties of $\mu(u, h)$ with relation to n-metacaloric functions.

F. W. Perkins (Hanover, N.H.).

Myškis, A. D.; and Šlopak, A. S. Mixed problem for systems of differential-functional equations with partial derivatives and Volterra type operators. Mat. Sb. N.S. 41(83) (1957), 239–256. (Russian)

In a previous paper Myškis has studied the continuous dependence of the solutions upon the data for a boundary problem for a pair of first-order partial differential equations $L_s(i, u)=h_s$, $(s=1, 2)$, $i=i(x, t)$ and $u=u(x, t)$ being vectors [Mat. Sb. N.S. 30(72) (1952), 317–328; MR 14, 51]. The present paper extends the investigation to the system $L_s(i, u)+T_s(i, u)=h_s$, where T_1, T_2 are "Volterra type operators". In the first group of results T_1 and T_2 have no explicit form, but have to satisfy certain integral inequalities; these bound in mean-square the derivatives of $T_s(i_1, u_1)-T_s(i_2, u_2)$ in terms of those of i_1-i_2, u_1-u_2. Under such conditions the authors estimate the variation in i, u and their derivatives resulting from changes in h_1,

h_2 and in the initial values of i, u. Variation in the coefficients is considered separately. The case in which the T_s have the usual Volterra form is then shown to be included, as is also a system considered by M. I. Rozovskiĭ, [Dokl. Akad. Nauk SSSR (N.S.) **59** (1948), 1265–1268; MR **9**, 594]. Finally the existence, uniqueness and properties of generalised solutions are established in the special case of the system

$$L\partial i/\partial t+\partial u/\partial x+Ri+\bar{T}_1(i)=h_1,$$
$$C\partial u/\partial t+\partial i/\partial x+Gu+\bar{T}_2(u)=h_2,$$

where the $\bar{T}_s$ are again not given explicitly, but merely satisfy certain boundedness and linearity conditions. This extends another investigation of Myškis [Mat. Sb. N.S. **31(73)** (1952), 335–352; MR **14**, 656]. *F. V. Atkinson.*

Eringen, A. Cemal. A transform technique for boundary-value problems in fourth-order partial differential equations. Quart. J. Math., Oxford Ser. (2) **6** (1955), 241–249.

Let $Lu=u^{(IV)}+(pu')'+qu$ and let u_n be the eigenfunctions of the self-adjoint system

$$Lu+\lambda u=0,$$
$$U_i=\sum_{j=1}^{4} a_{ij}u^{(j-1)}(a)=0,\quad \bar{U}_i=\sum_{j=1}^{4} b_{ij}u^{(j-1)}(b)=0\quad (i=1,2),$$

where a_{ij} and b_{ij} are constants and U_1 and U_2 as well as $\bar{U}_1$ and $\bar{U}_2$ are linearly independent. The finite T-transform of f is defined to be the sequence $f_n=T(f)=N_n^{-2}\int_a^b f(x)u_n(x)dx$, where N_n is the normalizing factor for u_n. The inverse transform is $f=\sum_n f_n u_n(x)$. It is shown that the solution of the partial differential equation $L_x v(x,y)=M_y v(x,y)$, where L is as defined above and $M_y v(x,y)=\sum_{i=0}^{n} a_i(y)\partial^i v/\partial y^i$, with boundary conditions imposed at $x=c_1$, $x=c_2$, $y=d_1$, $y=d_2$, can be reduced to the problem of solving a set of ordinary differential equations by application of the finite T-transform.

This method is applied to the solution of a vibrating beam problem arising in ram-jet valves.

F. Goodspeed (Vancouver, B.C.).

Ghizzetti, Aldo. Sulle equazioni alle derivate parziali del 2° ordine, lineari, in due variabili indipendenti, le cui soluzioni godono di proprietà integrali rispetto ad una delle variabili. Ann. Mat. Pura Appl. (4) **40** (1955), 41–59.

Given an equation of the form

$$E[u]\equiv a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+2p(x,y)u_x+2q(x,y)u_y+r(x,y)u=f(x,y)$$

in a domain T: $x_0\leqq x\leqq x_1$, $y_0(x)\leqq y\leqq y_1(x)$, it is sometimes possible to reduce this to an ordinary differential equation by means of an integral transformation, such as a Laplace or Fourier transformation, with respect to y. The purpose of this paper is to try to determine when this type of reduction is possible — i.e., when one can select a function $v(x,y)$ so that the equation

$$\int_{y_0(x)}^{y_1(x)} E[u]v(x,y)dy=\int_{y_0(x)}^{y_1(x)} f(x,y)v(x,y)dy$$

is equivalent to an ordinary differential equation in $U(x)=\int_{y_0(x)}^{y_1(x)} u(x,y)v(x,y)dy$, or a related function. After reducing the problem to canonical form, the author determines, under suitable additional hypotheses, just what combinations of coefficients will give the desired reduction; the results are too detailed to give here, since there are many cases and subcases. *D. L. Bernstein.*

Pucci, Carlo. Sui problemi di Cauchy non "ben posti". Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 473–477.

If $u(x,y)$ is a harmonic function in a region D containing the interval I: $0\leqq x\leqq 1$, $y=0$, in its interior, and if $u(x,0)=u_0(x)$ and $u_y(x,0)=u_1(x)$ for x on I, then one can find a harmonic function $v(x,y)$ such that $v(x,0)=v_0(x)$ and $v_y(x,0)=v_1(x)$ are arbitrarily close to u_0 and u_1 but such that $|u(x,y)-v(x,y)|$ is arbitrarily large at points of a neighborhood of I. For this reason the Cauchy problem for a harmonic function is said to be not well-posed. Nevertheless, in this paper, it is shown that if one can approximate the values of u and u_y at $4n+1$ equidistant points, for each n sufficiently large, and if the maximum error E_n at these points, as n becomes infinite, is $o(e^{-8nh})$, then one can construct a sequence $u_n(x,y)$ of functions which converge uniformly to $u(x,y)$ in a rectangle R: $0\leqq x\leqq 1$, $|y|\leqq h'$, where h' is a number smaller than h, the distance of I from the boundary of D. It is also shown that the derivatives of u can be approximated uniformly. The paper ends with some interesting observations on the Cauchy problem for sequences of harmonic functions.

D. L. Bernstein (Los Angeles, Calif.).

Lavrent'ev, M. M. On the Cauchy problem for the Laplace equation. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 819–842. (Russian)

In 2 or 3 dimensions let Cauchy data for Laplace's equation be given approximately on a portion S of the boundary of a region D in which a harmonic function u is known to exist; i.e., in some norm let continuous functions φ_n and ψ_n be given with $\|\varphi_n-u\|_S\leqq\alpha_n$, $\|\psi_n-\partial u/\partial n\|_S\leqq\beta_n$, where $\alpha_n\to 0$ and $\beta_n\to 0$ as $n\to\infty$. The problem is to find methods of "effective solution" of the Cauchy problem for u, i.e., formulas determing a sequence u_n from φ_n and ψ_n on S, such that $u_n\to u$ for each point of D. The paper is based on prior work of Carleman [Les fonctions quasi analytiques, Gauthier-Villars, Paris, 1926], Pucci [see the paper reviewed above] and Lavrent'ev [Dokl. Akad. Nauk SSSR (N.S.), **106** (1956), 389–390; MR **18**, 799].

The problem is reformulated several times in terms of analytic functions, and in different norms, and three solutions are given. Each of them is said to be "unstable" in the sense that, if no hypothesis is made on u, it is impossible to assert that $\|u_n-u\|\to 0$ with guaranteed speed. However, if it is assumed that $\|u\|\leqq M$ in D, then the methods are "stable", in that there are guaranteed bounds $\|u_n-u\|\leqq\gamma_n$, with $\gamma_n\to 0$ as $n\to\infty$. The first solution (2 dimensions) is in terms of Cauchy integrals over S. The second (2 dimensions) is preceded by a conformal transformation of D to a rectangle, followed by Fourier expansions of φ_n and ψ_n. The third is given for 3 dimensions. As tools the author proves several new theorems which are related to an inequality of Carleman [op. cit.], of which the following is typical. Let $u(x,y)$ be harmonic and regular in the strip $0\leqq y\leqq 1$, with $u(x+2\pi,y)=u(x,y)=-u(-x,y)$. Assume $u(x,0)=u(\pi,y)=u(0,y)=0$ and

$$\int_0^\pi\{u(x,1)\}^2dx\leqq M^2,\quad \int_0^\pi\{u(x,0)\}^2dx\leqq m^2.$$

Then

$$\int_0^\pi\{u(x,y)\}^2dx\leqq M^{2y}\cdot m^{2(1-y)}\cdot y^2.$$

G. E. Forsythe (Stanford, Calif.).

See also: Harmonic Functions, Convex Functions: Morrey and Eells; Morrey. **Integral Transforms:** Widder

(2 articles). **Ordinary Differential Equations:** Stepanow; Pailloux. **Integral and Integrodifferential Equations:** Cameron. **Numerical Methods:** Witting; Freeman and Baer. **Fluid Mechanics, Acoustics:** Morawetz. **Quantum Mechanics:** Kato.

Difference Equations, Functional Equations

★**Montel, Paul. Leçons sur les récurrences et leurs applications. Recueillies et rédigées par Jacques Dufresnoy et Éloi Lefebvre.** Gauthier-Villars, Paris, 1957. xi+268 pp. 4300 fr.

This is the latest volume of the Borel Monographs on the Theory of Functions, and was completed about six months after the death of Borel on February 3, 1956. It is a remarkable work. It will delight both the general reader and the specialist to see how the author has managed to crowd into a book of this size a mass of beautifully organized material, incorporating many of the author's own contributions.

In Chapter I, on Definitions and Generalities, the author begins with recurrences, defining the arithmetic and geometric progressions. He also gives A. Hurwitz's definition of the natural logarithm by means of the recurrence $u_{n+1}=\sqrt{u_n}$. Difference equations are introduced and an application is made to the characterization of central conics. The notion of a general recurrence of the pth order and its associated generating function are defined. As he later shows, the nature of the recurrence is closely related to the distribution of the singularities of the generating function.

Chapter II is devoted to the study of linear recurrences and, in particular, to the linear difference equation with periodic coefficients. This paves the way for the introduction of the determinant of Casorati which is the analogue of the Wronskian for linear differential equations. A digression is made here on determinants and their adjoints to solve the problem of finding the difference equation of minimum order satisfied by the given recurrence. Several interesting applications are made of the operator method. He shows, for example, that a necessary and sufficient condition for a permanently convergent series to represent a rational function is that the coefficients of the series satisfy a linear recurrence with constant coefficients. This has its analogue in the theory of integral equations [Lalesco, Introduction à la théorie des équations intégrales, Gauthier-Villars, Paris, 1912, pp. 38–42]. Another application concerns trigonometric series. He proves that a necessary and sufficient condition for the series $u_0+\sum_{n=1}^{\infty}(u_n\cos nx+v_n\sin nx)$ to represent a rational function of $\cos x$ and $\sin x$ is that the coefficients u_n and v_n satisfy the same linear recurrence with constant coefficients such that the characteristic roots all have moduli less than unity.

Chapter III treats recurrences of the first order, $u_{n+1}=f(u_n)$. The values α satisfying $u=f(u)$ define the double points, and $f'(\alpha)$ is called the multiplier of the recurrence; if $|f'(\alpha)|<1$, then the double point is said to be attractive. The author shows that if $F(x)=\sum_{n=0}^{\infty}u_nx^n$ is the generating function corresponding to the recurrence $u_{n+1}=f(u_n)$, where f is a holomorphic function in a neighborhood of an attractive double point with non-zero multiplier, then F is the quotient of two entire functions of genus zero. A further study of the generating function is made by its relationship to the functional equation of Schröder $F[f(u)]=sF(u)$, where s is the multiplier relative to the double point in question.

Chapter IV deals with recurrences of higher order. The relation between the recurrence and the functional equation of Schröder, which now contains several variables, is studied. The proof of the existence of the functions of Schröder depends on the existence of invariant curves under a point transformation, a subject already treated by Hadamard and Lattès. An application to trigonometric series, whose coefficients are connected by certain recurrence relations, is made.

In order to show the close analogy that exists between recurrences and differential equations, the author devotes Chapter V to a proof of a theorem due to Poincaré. The problem is to study the limit of the ratio u_{n+1}/u_n of a linear recurrence as $n\to\infty$. This ratio may tend to the root of maximum modulus of a certain characteristic equation associated with the recurrence. In case the ratio tends to one of the other roots, the solutions of a certain differential equation satisfied by the corresponding generating functions, are holomorphic in a neighborhood of the root of maximum modulus.

Chapters VI–IX, almost half of the book, are concerned with the theory of continued fractions. This is natural since continued fractions have as their basis linear recurrences. However, here, they are studied for the most part from the point of view of their analytical, rather than their algebraic or arithmetical properties. Chapters VI and VII deal with finite and infinite continued fractions, respectively. Chapter VIII has to do with the convergence of continued fractions, first with real, then with complex terms. The tests of Pringsheim, Jacobi, Perron, and Van Vleck are mentioned.

In Chapter IX the author considers particular continued fractions F. He defines certain permanently convergent series in z related to F and calls them the correspondents and associates of F. Applications of these notions are made to Bessel functions and to the development of the square root of a polynomial as a continued fraction.

Chapter X is devoted to the study of Stieltjes fractions. In Chapter IX the author utilized the correspondent and associate continued fractions relative to an element at the origin to obtain the analytic continuation of the function defined by this element. In this chapter he employs these fractions to obtain the analytic continuation of a function defined by an element at infinity. The author shows that relative to this element there exist a correspondent and associate and these are the Stieltjes fractions. Stieltjes integrals and moments are used throughout.

In Chapter XI the author resumes the study of iteration begun in Chapter III. He now considers properties which are no longer local but general. Classifications of fixed and double points are given. A particularly interesting result is: a recurrence defined by the rational function $P(z)/Q(z)$, where P and Q are of the same degree greater than one in z, yields cycles of as high an order as desired. While the names of Koenings, Grévy, Fatou, and Julia are mentioned, it is surprising to find no reference to J. F. Ritt, the American mathematician.

In Chapter XII properties of rational functions having a fundamental circle are considered. This is a class of functions which take the real axis into itself and preserve or interchange the complex half-plane. The chapter ends with the study of meromorphic functions which are (uniform) limits of sequences of rational functions with fundamental circles.

It is seen from this resumé that the book contains

material which cuts across many fields of mathematics and which is of interest to all mathematicians. However, the exposition flows along in narrative form, as is usual in the writings of the French School. Rarely are the precise assumptions given, even in the italicized statement of a result. The reader would have trouble in putting his finger on any particular theorem without reading the material relative to it. While there is a complete table of contents, there is no index. This is a real deficiency in view of the many new terms used. The misprints are very few in number and may be detected readily. The reviewer believes that an English translation of this work, with appropriate references, would be a valuable addition to the literature in this field. *I. A. Barnett.*

Bellman, Richard. Functional equations in the theory of dynamic programming. III. Rend. Circ. Mat. Palermo (2) **5** (1956), 297–319 (1957).

[For parts I–II see Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1077–1082; **41** (1955), 482–485; MR **15**, 887; **17**, 264.] This paper contains existence, unicity and stability theorems and a construction by successive approximation of the solution of the functional equation

$$f(P, P')=\max_{G}\min_{G'}\Big\{\iint_{D(P,P')}[R(P, P'; u, v)+$$

$$h(P, P'; u, v)f\{T(P, P'; u, v), T'(P, P'; u, v)\}]dG(u)dG'(v)\Big\}$$

with the simultaneous equation arising by changing $\max_G \min_{G'}$ into $\min_{G'} \max_G$, and of similar equations. The author gives a clear and detailed account of how this equation is derived and applied to multi-stage games by the principle of optimality from the theory of dynamic programming. Some misprints: On p. 229, l. 21, replace the second $R(u, v)$ by $h(u, v)$; p. 305, l. 10, replace $S(S'(P, P')$ by $S'(P, P')$; and on p. 306, l. 2, replace T by T'. *J. Aczél* (Debrecen).

See also: Special Functions: Aissen. **Integral and Integrodifferential Equations:** Cameron.

Integral and Integrodifferential Equations

★ **Mikhlin, S. G. Integral equations and their applications to certain problems in mechanics, mathematical physics and technology. Translated from the Russian by A. H. Armstrong.** Pergamon Press, New York-London-Paris-Los Angeles, 1957. xii+338 pp. $12.50.

Translation of the second Russian edition [MR **12**, 712].

Cameron, R. H. Nonlinear Volterra functional equations and linear parabolic differential systems. J. Analyse Math. **5** (1956/57), 136–182.

In the paper under review the author shows that an equivalence holds between almost everywhere existence problems for a very general class of non-linear integral equations, minimality (or uniqueness) problems for certain linear parabolic differential systems in several variables, and problems concerning the evaluation of a general class of Wiener integrals. The non-linear integral or functional equations considered in general involve several integrals. For example when there are just two, the general form considered is

$$y(t)=x(t)+\int_0^t P\{s, x(s), \int_0^s Q[r, x(r)]dr\}ds.$$

Calling the second term on the right a "Volterra functional of order 2", the author studies Volterra functional equations of any finite order. The first main theorem is an (almost everywhere) existence theorem for non-linear Volterra functional equations. In the proof of this theorem the author uses among other things a transformation theorem on Wiener integrals developed by the author and Fagen [Trans. Amer. Math. Soc. **75** (1953), 552–575; MR **15**, 536]. Having this first main theorem the author uses it to develop (as his Theorem 2) a method for the evaluation of a class of Wiener functionals. He then restates this theorem from the point of view of stressing differential systems and this enables him to obtain sufficient conditions that a positive solution be minimal. The paper continues with the development of various further theorems. As the author states near the end of the paper, "A comparison of the theorems proved up to this point shows that there is a certain duality between almost everywhere existence problems for solutions of non-linear Volterra functional equations, and minimality problems for certain linear partial differential systems." He then states and proves two theorems "which present the duality theorem from opposite points of view" and then closes with a consideration of the three equivalences mentioned earlier applied to a special case of interest. *W. T. Martin* (Cambridge, Mass.).

Kohn, J. J.; and Spencer, D. C. Complex Neumann problems. Ann. of Math. (2) **66** (1957), 89–140.

A mineographed form of this paper [Complex Neumann problems on Kähler manifolds, Princeton University, Princeton, N.J., 1956] was reviewed in MR **18**, 658.

Melzak, Z. A. A scalar transport equation. Trans. Amer. Math. Soc. **85** (1957), 547–560.

This paper is concerned with the existence and uniqueness of the solutions of the following equation

$$\frac{\partial f}{\partial t}=[f(x, t), f(x, t)]+Lf(x, t), \tag{1}$$

where $[f, g]$ and Lf are given by

$$[f, g]=\tfrac{1}{2}\Big\{\int_0^x f(y)g(x-y)\phi(y, x-y)dy$$

$$-f(x)\int_0^\infty g(y)\phi(x, y)dy-g(x)\int_0^\infty f(y)\phi(x, y)dy\Big\},$$

$$Lf=\int_x^\infty f(y)\psi(y, x)dy-\frac{f(x)}{x}\int_0^x y\psi(x, y)dy.$$

Here the variables x, y, t are non-negative and the functions $f(x, 0)$, $\phi(x, y)$ and $\psi(x, y)$ are assumed to be known. The functions ϕ and ψ are assumed to satisfy the following hypotheses: (H_1) $\phi(x, y)$ is continuous,

$$0\leqq\phi(x, y)=\phi(y, x)\leqq\sup\phi(x, y)=A<\infty;$$

(H_2) $\psi(x, y)$ is continuous,

$$0\leqq\psi(x, y)\leqq\sup\psi(x, y)=C<\infty,$$

$$\int_0^x y\psi(x, y)dy\leqq x, \int_0^x \psi(x, y)dy\leqq E-1<\infty.$$

The proof of uniqueness depends on the following key lemma: Let $f(x, 0)$ be a continuous, non-negative, bounded and integrable function. Let $0<\sup f(x, 0)=B<\infty$ and $\int_0^\infty f(x, 0)dx=N<\infty$. Let $\phi(x, y)$ and $\psi(x, y)$ satisfy hypotheses (H_1) and (H_2). Then there exists a number $m=m(A, B, C, N, E)$, $0<m<\infty$, such that (1) possesses a solution $f(x, t)$ valid on the interval $0\leqq t<1/m$. This so-

lution is continuous, analytic in t for each x, and integrable in x for each t. The proof of this lemma consists in constructing a solution. By direct substitution it can be verified that the series (2) $f(x, t)=\sum_{k=0}^{\infty} a_k(x)t^k$ represents a formal solution of the equation, where

$$a_0(x)=f(x, 0),\ a_{k+1}(x)=\frac{1}{k+1}\left\{\sum_{i+j=k}[a_i,\ a_j]+La_k\right\}.$$

The author shows that the series (2) is uniformly convergent on the interval $0\leqq t<1/m$ and satisfies all the conditions of the lemma. This solution is then continued for $t<1/2m$ and successively to larger values of t; and the uniqueness of the solution is established by showing that the local existence intervals cover the whole interval $0\leqq t<\infty$. *S. Chandrasekhar.*

See also: Ordinary Differential Equations: Hebroni. **Fluid Mechanics, Acoustics:** Littman. **Quantum Mechanics:** Gourdin and Martin (2 articles); Drukarev.

Calculus of Variations

Schäfke, Friedrich Wilhelm. Bemerkungen über Variationsrechnung, implizite Funktionen und Lagrangesche Multiplikatoren. Math. Z. **67** (1957), 252–266.

In the calculus of variations theory for a simple integral $\int_{t_1}^{t_2} L(t, x, x')dt$, it is sometimes desirable to omit the assumption that L is differentiable with respect to t. The author remarks that this omission can be justified, even for the Hamilton-Jacobi theory, by using a modified implicit theorem in which differentiability with respect to some of the independent variables is not required. He states and proves such a theorem in terms of Banach spaces. It is a "theorem in the large" in the special sense that the solution is defined in a neighborhood of a given compact set. The proof of the local theorem given by Hildebrandt and Graves [Trans. Amer. Math. Soc. **29** (1927), 127–153, p. 150] can be modified in an obvious way to give the author's result "in the small". However, the author's proof of the theorem in the large is probably easier to read than that given by Graves for the case when there is differentiability with respect to all variables [ibid. **29** (1927), 514–552, p. 532]. The author makes another application of the theorem to derive a multiplier rule for a minimum problem in a Banach space, with side conditions. His assumptions are in some respects more restrictive than those of Goldstine [Bull. Amer. Math. Soc. **44** (1938), 388–394; **46** (1940), 142–149; MR **1**, 146]; in particular he assumes the existence of a projection of the space of the independent variables on a subspace.

L. M. Graves (Chicago, Ill.).

See also: Probability: Dvoretzky; Rustagi.

TOPOLOGICAL ALGEBRAIC STUCTURES

Wallace, A. D. The peripheral character of central elements of a lattice. Proc. Amer. Math. Soc. **8** (1957), 596–597.

The author shows that the central elements in a compact connected lattice are contained in the cohomological boundary. The proof is based on a previous result of the author [Nederl. Akad. Wetensch. Proc. Ser. A. **59**= Indag. Math. **18** (1956), 271–274; MR **18**, 14].

A. Shields (Ann Arbor, Mich.).

Gil de Lamadrid, J.; and Jans, J. P. Note on connectedness in topological rings. Proc. Amer. Math. Soc. **8** (1957), 441–442.

The authors prove the following result. If a ring A admits a simply connected covering space (S, f), then S can be made into a ring so that f is a ring homeomorphism. Moreover, if A has an identity, $S=A$. The proof is analogous to the corresponding result for groups together with the observation that a discrete ideal in a connected ring with identity must consist of a single point.

P. S. Mostert (New Orleans, La.).

See also: Ordinary Differential Equations: Hebroni. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Širokov; Kleinecke.

Topological Groups

Pettis, B. J. On a vector space construction by Hausdorff. Proc. Amer. Math. Soc. **8** (1957), 611–616.

Hausdorff proved [J. Reine Angew. Math. **167** (1932), 294–311] that each infinite-dimensional Banach space contains a dense linear subspace of second category. The author formulates Hausdorff's construction in a much more general setting, and obtains a result which applies in E^n to produce a subgroup L which is impressively "pathological" relative to measure, category, and linear structure. Some of L's properties are as follows: (a) both L and $E^n\sim L$ have inner measure equal to zero; (b) if a Baire set lies in L or in $E^n\sim L$, it is of first category; (c) L is the kernel of an open additive discontinuous map of E^n onto the rationals; (d) L is the image of E^n under an open additive discontinuous projection (and of course L is dense in E^n, of second category, and not a Baire set). This strengthens earlier results of Kodaira [Proc. Phys.-Math. Soc. Japan (2) **23** (1941), 67–119; MR **2**, 317] and Halperin [Proc. Amer. Math. Soc. **2** (1951), 221–224; MR **12**, 685]; part of the present proof is a paraphrase of one by Halperin. Some other applications of the author's main theorem are also discussed.

{Reviewer's comments: 1) The remark by Erdös referred to in the last sentence of the paper is as follows: Let E be an infinite-dimensional linear space, B a Hamel basis for E, and X a countably infinite subset of B. For each $x\in X$, let H_x be the linear extension of $B\sim\{x\}$. Then $\bigcup_{x\in X}H_x=E$, and hence E is the union of countably many maximal proper subspaces. 2) The following amusing consequence of the author's results may be worth mentioning: Let E be a second category topological linear space, $\mathfrak{M}$ the set of all maximal proper subspaces of E, and $\mathfrak{D}$ the set of all members of $\mathfrak{M}$ which are dense in E. Then $\mathfrak{M}$ is "algebraically close" to $\mathfrak{D}$ in the following sense: if $M\in\mathfrak{M}$ and B is a Hamel basis for M, there exist $b\in B$ and $b'\in E$ such that the linear extensions of $(B\sim\{b\})\cup\{b'\}$ is a member of $\mathfrak{D}$. (In general, of course, $\mathfrak{M}$ is not similarly "close" to the set of closed members of $\mathfrak{M}$.)}

V. L. Klee, Jr. (Seattle, Wash.).

Hartman, S.; et Hulanicki, A. Les sous-groupes purs et leurs duals. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 141, XIII. (Russian summary).

Let G be a locally compact abelian group, G^* its dual, H a closed subgroup, A the annihilator of H. H is called

pure if and only if $nH=H\cap nG$ for all integers n. H has property P if and only if each character of finite order of H can be extended to a character of G of the same order. Three theorems are stated. The first says A is pure if and only if H has property P. Let K be the smallest class of such G which are generated by a compact neighborhood of the identity and closed under duality. The other theorems state: If H is pure and $G\in K$, then A is closed in G^*. If $G\in K$, then H is pure if and only if it has property P.

W. Ambrose (Cambridge, Mass.).

Hartman, S.; und Ryll-Nardzewski, C. Zur Theorie der lokal-kompakten Abelschen Gruppen. Colloq. Math. **4** (1957), 157–188.

An expository paper on various aspects of the theory of locally compact abelian groups. The following subjects are discussed: 1) ergodic elements in compact abelian groups; 2) connectivity and equivalent properties for compact abelian groups; 3) the connected component of the identity as direct summand; 4) discussion of the problem in 2) for locally compact abelian groups; 5) properties of the set of "quotients" a/n $(n=1, 2, \cdots)$ of an element a in a connected compact (abelian) group; 6) topologies which make the additive group of real numbers into compact abelian groups; and 7) torsion-free almost periodic functions. In particular, an example of a locally compact abelian group G is given in 4) such that G is not divisible, but $\chi(G)$ is infinite for any non-trivial character χ of G.

K. Iwasawa (Cambridge, Mass.).

Anderson, R. D. Zero-dimensional compact groups of homeomorphisms. Pacific J. Math. **7** (1957), 797–810.

The author constructs a number of topological transformation groups (G, M) in which G acts on M without fixed points, i.e., $gx\neq x$ for every $x\in M$ and $g\in G-1$. If M is the universal curve (i.e., locally connected continuum in which every 1-dimensional continuum is embeddable), then any compact 0-dimensional G can be made to act on M without fixed points so that M/G is homeomorphic to M. If M is the universal curve, any infinite compact 0-dimensional G can be made to act on M without fixed points so that M/G is a regular curve (one in which any two points are separated by finite sets). If G is finite, there exists in E^3 a 3-manifold with connected boundary, on which G can be made to act without fixed points.

P. A. Smith (New York, N.Y.).

Koosis, Paul. An irreducible unitary representation of a compact group is finite dimensional. Proc. Amer. Math. Soc. **8** (1957), 712–715.

The author gives a direct elementary proof of the theorem in the title by showing that the square of a certain self-adjoint operator obtained by an integration over the representation operators is both completely continuous and equal to a constant times the identity.

L. H. Loomis (Cambridge, Mass.).

See also: Groups and Generalizations: Banaschewski. **Trigonometric Series and Integrals:** Domar. **Topological Vector Spaces:** Pettis.

Lie Groups and Algebras

Tits, J. Les groupes de Lie exceptionnels et leur interprétation géométrique. Bull. Soc. Math. Belg. **8** (1956), 48–81.

In this paper, the author gives a unified geometrical interpretation for all complex semisimple Lie groups. To each complex semisimple Lie group G, he associates a geometry Γ which consists of (i) a set of fundamental elements divided into a finite number of families which are in one-to-one canonical correspondence with the vertices of the diagram of Witt-Dykin (i.e., diagram of Schläfli) of G, and (ii) an incidence relation defined on the set. In fact, these fundamental elements can be regarded as the maximal non-semisimple subgroups of G. They are divided into n families by inner automorphisms of G, where $n=\text{rank}(G)$. Two such subgroups K_1, K_2 are said to be incident if $K_1\cap K_2$ contains a maximal solvable subgroup of G. Many general properties of Γ are established. When G is the simple group of the class A_n, then the families $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_n$ of fundamental elements are respectively, the families of points, lines, $\cdots$, $(n-1)$-planes in the complex projective space P_n, and the incidence relation is the inclusion. Thus the geometry Γ associated with A_n is the n-dimensional projective geometry.

By an automorphism of Γ, the author means a permutation of the set of fundamental elements carrying each family into itself and preserving the incidence relation. Every family of fundamental elements has a natural complex analytic structure, and hence continuous and analytic automorphims can be defined. The main result can be stated as follows. Let G be a complex semisimple Lie group without centre, and Γ its associated geometry; denote by G_t, G_a, G_c, respectively, the group of all automorphisms, analytic automorphisms, continuous automorphisms of Γ, and suppose that G does not contain the simple group A_1 of rank one as a factor. Then G_a is isomorphic with G, and is an invariant subgroup of G_t. Moreover, G_c/G_a is of order two, and G_t/G_c is isomorphic with the group of automorphisms of the field of complex numbers.

H. C. Wang (Evanston, Ill.).

Takenouchi, Osamu. Families of unitary operators defined on groups. Math. J. Okayama Univ. **6** (1957), 171–179.

Let H be the group of all 3×3 real matrices with ones on the main diagonal and zeros below it. Let G be a simply connected solvable Lie group which contains H as a normal subgroup and whose center contains the center of H. The author shows that there exists an essentially unique complex valued function α on $(G/H)\times(G/H)$ such that determining the non-obvious irreducible unitary representations of G is equivalent to determining the projective unitary representations on G/H with multiplier α and hence to determining the ordinary unitary representations of a group having dimension one higher than that of G.

G. W. Mackey (Cambridge, Mass.).

Ehrenpreis, Leon. Some properties of distributions on Lie groups. Pacific J. Math. **6** (1956), 591–605.

Let G be a Lie group with a countable set of connected components and V a topological vector space of the Fréchet type. Denote by $\mathscr{E}$ the set of (indefinitely) differentiable mappings $G\to V$, and by $\mathscr{D}$ the space of functions in $\mathscr{E}$ with compact carriers, these spaces being topologized in the usual manner. The spaces $\mathscr{D}$ and $\mathscr{E}$ are reflexive Montel spaces; let $\mathscr{D}'$ and $\mathscr{E}'$ be their dual spaces. The author defines an operation of convolution $(S, f)\to S*f$ between elements $S\in\mathscr{D}'$ and $f\in\mathscr{D}$, with values in the space $\mathscr{E}(C)$ of differentiable complex-valued functions on G. He shows that, given an S in $\mathscr{D}'$, a necessary and sufficient condition for S to be in $\mathscr{E}'$ (i.e., to have a compact carrier) is that $S*f$ be in $\mathscr{D}(C)$ for all f in $\mathscr{D}$; moreover, if we assign to every S in $\mathscr{E}'$ the mapping $f\to S*f$, we

obtain an isomorphism of $\mathscr{E}'$ with a subspace of the space of linear mappings $\mathscr{D} \to \mathscr{D}(C)$ (the latter space being given the compact-open topology). These results are partially extended to the case where V is not assumed to be of the Fréchet type. The last section is concerned with the case where V is the space of endomorphisms of a space X and there are given representations of two compact subgroups of G on the space V; however, the meaning of Theorem 3 of the paper is not clear, because values of convolutions are complex-valued functions and the meaning of such symbols as ${}_{UW}\mathscr{D}(C)$ is not specified. *C. Chevalley.*

Trifonov, E. D. Irreducible representations of a four-dimensional rotation group. Vestnik Leningrad. Univ. 12 (1957), no. 4, 25–30. (Russian)

Motivated by quantum-mechanical applications, the author determines in notably explicit form the reduction into irreducible constituents of Kronecker products of irreducible unitary representations of the orthogonal group in four dimensions. *I. E. Segal* (Chicago, Ill.).

Graev, M. I. Unitary representations of real simple Lie groups. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 179–182. (Russian)

A sketch of progress in the last five years in the determination of the irreducible unitary representations of real simple groups, particularly those whose complex extension is the complex unimodular group, and also in the derivation of Plancherel's formula for such groups.

I. E. Segal (Chicago, Ill.).

★**Matsushima, Yozô. Sur les algèbres de Lie linéaires semi-involutives.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 17 pp.

In his work on infinite transformation groups, E. Cartan has introduced the notion of semi-involutive Lie algebra, and determined all complex irreducible Lie algebras of this type. The present paper solves the corresponding problem for the real case. The author gives a careful discussion of the concepts involved, such as "the vector space $L^{(1)}$ deduced from a space L of endomorphisms of a vector space V (of dimension n)": Let $B_1(L)$ consist of the symmetric bilinear maps b from $V \times V$ to V such that the maps $b(v)$, obtained by fixing one variable in b, belong to L. Each b determines an endomorphism $h(b)$ of $V \times L$ by $h(b)(v, a) = (0, b(v))$; the $h(b)$ form $L^{(1)}$. One calls L ($\neq 0$) involutive if $L^{(1)}$ is of maximal dimension in a certain sense; L is semi-involutive if some deduced $L^{(k)}$ is involutive; L is of finite type if some deduced $L^{(k)}$ is zero. The irreducible, semi-involutive algebras are the following: $\mathrm{gl}(V)$, $\mathrm{sl}(V)$, $\mathrm{sp}(V)$, the general, special and symplectic algebra; $\mathrm{sp}(V)+Z$, where Z is the center of $\mathrm{gl}(V)$; $\mathrm{sl}(W)+U$, where W is a complex vector space and U a real subspace of the center of $\mathrm{gl}(W)$; $\mathrm{sp}(W)+U$, with W and U as before. In addition, the (eight) classes of irreducible Lie algebras L are determined whose deduced $L^{(1)}$ is non-trivial. *H. Samelson* (Ann Arbor, Mich.).

★**Séminaire "Sophus Lie" de la Faculté des Sciences de Paris, 1955-56. Hyperalgèbres et groupes de Lie formels.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1957. 61 pp. (multigraphiées)

During the last two years, P. Cartier has systematically simplified, extended and developed the theory of formal Lie groups and of Lie hyperalgebras over an arbitrary field, which was started by the reviewer in 1953; he has also obtained important applications of that theory to algebraic groups and has recently announced in two notes some of his most significant results [C. R. Acad. Sci. Paris 244 (1957), 426–428, 540–542; MR 18, 870, 789]. Some of these results were the subject of the "Sophus Lie" Seminar of 1955–56, which was conducted by Cartier and M. Lazard; the volume under review, however, only covers the first half of the seminar; the rest will be incorporated in Cartier's thesis. Exposé 1 gives a very general definition of "varieties" à la Serre covering differential and analytic manifolds as well as algebric varieties; "punctual distributions" are introduced (i.e., differential operators of all orders at a given point; the usual "tangent vectors" are the "distributions of order 1"), and it is then shown how a group law on the "variety" gives rise to a structure of Lie hyperalgebra on the distributions at the origin. This gives the motivation for the abstract study of the structure of hyperalgebra, which occupies the remainder of the work. Exposé 2 gives the definition of the general notion of hyperalgebra introduced by Cartier, which contains the Lie hyperalgebras as special cases (see for that definition the review of the second note mentioned above); the elementary consequences of the definition are derived, and it is shown how on the dual of a hyperalgebra the "transposed" structure gives rise to a structure of commutative algebra, together with a "diagonal mapping" (in the Lie case, this algebra is the algebra of formal power series, and the diagonal mapping is a group law). Exposé 3 introduces some simple examples of hyperalgebras, namely the universal enveloping algebras of Lie algebras and of Lie-p-algebras ("restricted" Lie algebras in the terminology of Jacobson), and the hyperalgebra $\Gamma(M)$ of "divided powers" on a module M (which, when M is free, turns out to be the Lie hyperalgebra of a (finite of infinite) direct sum of additive groups); this chapter is noteworthy for its particularly careful and elegant treatment of Lie-p-algebras and their enveloping algebras. Exposés 4 and 5, which end the volume, are the most original ones; they inaugurate the cohomological study of hyperalgebras, and more generally of "coalgebras", a coalgebra being essentially the dual of an associative algebra, with a "diagonal mapping" which results by transposition from the multiplication law. As a preliminary, modules over a coalgebra are defined in a natural way; then, according to the general Cartan-Eilenberg method, it is shown that any such module admits of an "acyclic and $\mathscr{E}$-injective resolution". The road is then open for the application of standard techniques of homological algebra, allowing one to define the functors $\mathrm{Ext}_A^i(M, N)$ for any two modules M, N on a coalgebra A, and in particular the cohomology groups $H^i(A) = \mathrm{Ext}_A^i(K, K)$, where K is the ring of scalars of A (which can naturally be considered as an A-module); a "standard complex" is then defined, which enables one to compute more easily the $H^i(A)$. As an application, it is shown that the cohomology ring of $\Gamma(M)$, when M is free, is the exterior algebra $\Lambda(M)$. *J. Dieudonné.*

See also: Algebras: Gurevič; Karpelevič. **Groups and Generalizations:** Steinberg. **Topological Vector Spaces:** Williamson. **Manifolds, Connections:** Singh and Mishra; Frankel; Kostant.

Topological Vector Spaces

Williamson, J. H. A third condition equivalent to normability. J. London Math. Soc. 32 (1957), 231–232.

The present note is a sequel to the article in the same J. 31 (1956), 111–113 [MR 17, 1111]. The question raised

here is whether, in the algebra of endomorphisms of a vector space with a reasonable Hausdorff topology, the continuity of the bracket operation $(U, V) \to [U, V] = UV - VU$ implies the normability of the vector space. This is of interest in connection with the theory of Lie algebras. This note contains an affirmative answer to the above question. *From the introduction.*

Phelps, R. R. Convex sets and nearest points. Proc. Amer. Math. Soc. **8** (1957), 790–797.

For a closed set S in E^n denote by S_x the set all points which have x as nearest point on S. For convex S this set S_x is the complementary cone to the supporting cone of S at x and hence convex. It is shown that S is convex when S_x is a cone for every x on the boundary of S. The same holds for any normed linear space E with smooth sphere. E is an inner-product space if S_x is convex for each (not necessarily convex) S. Other characterizations of inner product spaces of dimension ≥ 3 follow: for instance, by the convexity of S_x for convex S. These are, however, merely reformulations of a result of Blaschke [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **68** (1916), 50–55]. Similarly, the results on symmetric orthogonality in two dimensions are contained in J. Radon, ibid. **68** (1916), 23–28.

These well-known papers have been consistently overlooked by the Banach space people and their results are attributed to papers which appeared in 1947 or even later.

H. Busemann (Los Angeles, Calif.).

Pettis, B. J. Comments on open homomorphisms. Proc. Amer. Math. Soc. **8** (1957), 583–586.

The author discusses openness of homomorphisms in linear topological spaces and metric groups, improving earlier results by various authors and introducing some useful new methods for attacking such problems. Let $\mathfrak{L}$ be the class of all locally convex linear topological spaces and $\mathfrak{A}$ a function on $\mathfrak{L}$ which associates with each $X \in \mathfrak{L}$ a class $\mathfrak{A}(X)$ of closed convex circled subsets of X. Suppose $\mathfrak{A}$ satisfies the following conditions: for each $X \in \mathfrak{L}$, the family of polars in X^* of the members of $\mathfrak{A}(X)$ is a local base for a topology τX in X^* such that $(X^*, \tau X) \in \mathfrak{L}$ and $(X^*, \tau X)^* = X$; if $X \in \mathfrak{L}$ and Y is a closed subspace of X, then $\mathfrak{A}(Y) = \{A \cap Y : A \in \mathfrak{A}(X)\}$. Under these assumptions on $\mathfrak{A}$, the author's first theorem asserts that if $X \in \mathfrak{L}$, Y is a closed subspace of X, and i is the injection map of Y into X, then the adjoint map i^* is a continuous open homomorphism of $(X^*, \tau X)$ into $(Y^*, \tau Y)$. This includes earlier results of Dieudonné and Munroe which apply only when $\mathfrak{A}(X)$ is the class of finite-dimensional (or compact) closed convex circled subsets of X. A corollary of the author's second theorem asserts that if X and Y are separable complete metric groups and h is a homomorphism of X onto Y, then each of the following implies the other two: h is continuous, h has a closed graph, h is open.

V. L. Klee, Jr. (Seattle, Wash.).

Deprit, André. Quelques classes d'homomorphismes d'espaces localement convexes séparés. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 252–272.

The author considers continuous linear mappings T of a locally convex linear topological space E into another such space F which are homomorphisms (i.e., $E/T^{-1}(0)$ is canonically bi-continuously isomorphic to $T(E)$). For such T where $T^{-1}(0)$ is finite-dimensional or $T(E)$ has finite deficiency in F the author develops a theory parallel to the treatment given by the reviewer in case E and F are Banach spaces [Duke Math. J. **18** (1951), 599–612; MR **13**, 355].

B. Yood (Eugene, Ore.).

Sebastião e Silva, José. Sui funzionali che sono funzioni di funzionali lineari dei loro argomenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 172–178.

Let S and E be linear spaces over the complex field and let D be a finitely open subset of S; that is, suppose that every finite-dimensional linear variety V in S meets D in a set open in V. If E is locally convex, a function F defined from D to E is called G-analytic if for each u in D and h in S the function $g(\lambda) = F(u + \lambda h)$ is analytic in a neighborhood of zero. Then $\lim_{\lambda \to 0} (F(u+\lambda h) - F(u))/\lambda$ is for each u in D a linear function of h, defined from S into E and denoted by $F'(u)$. Theorem 1: Let E be the complex field C, let F be a G-analytic function in D, and let $L_1, L_2, \cdots, L_n$ be linearly independent linear functionals defined on S. In order that there be a complex-valued function f of n complex variables such that for each u in D $F(u) = f(L_1(u), L_2(u), \cdots, L_n(u))$, it is necessary and sufficient that for each u in D the derivative $F'(u)$ be a linear combination of the $L_i(u)$, $F'(u) = \varphi_1(u)L_1(u) + \cdots + \varphi_n(u)L_n(u)$. Then the set $A = \{z : z = (L_1(u), \cdots, L_n(u))$ and $u \in D\}$ is open in C^n and f is holomorphic there.

From this the writer derives a theorem of Fantappiè [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) **3** (1928–29), 453–683] on analytic addition formulas for F. Theorem 2 of this paper is an analogue of Theorem 1 for the case where E is a space of holomorphic functions defined in a domain of the complex sphere.

M. M. Day (Urbana, Ill.).

★**Fan, Ky. On systems of linear inequalities.** Linear inequalities and related systems, pp. 99–156. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

This is the first theory of linear inequalities not restricted to finite-dimensional real space. Its scope can be seen from the following selection.

In Part I, the classical transposition theorems (incompatibility and implication conditions) for finite systems of linear inequalities are established for finite systems (1) $f_i(x) \geqq \alpha_i$, where the $f_i(x)$ are linear functionals on an arbitrary real linear space X. From Part II, where X is supposed finite-dimensional, we mention: (1) is inconsistent if and only if for some $i_1, \cdots, i_q$, for which $f_{i_1}, \cdots, f_{i_q}$ are dependent, and some $j_1, \cdots, j_{q-1}$, the corresponding coefficient submatrix F has $(q-1)$-minors F_k for which $F_k F_{k+1} < 0$ $(k = 1, \cdots, q-1)$, and $\sum_{k=1}^{q} \alpha_{i_k} |F_k| > 0$. Part III deals with Hilbert space and normed or topological linear spaces, including linear transformations, convex cones, and a new proof of the minimax theorem for a real continuous function on the product of two compact Hausdorff spaces. In Part IV, on linear spaces which may be real or complex, it is shown (in properly restricted spaces) that either $|f_j(x) - \alpha_j| \leqq r_j$ has a solution x or else $\sum \lambda_j f_j = 0$, $|\sum \lambda_j \alpha_j| \leqq \sum |\lambda_j| r_j$ has a solution λ; and that either $|f(x_\nu) - \alpha_\nu| \leqq r_\nu$ (not necessarily finite) has a solution f, $\|f\| \leqq \rho$, or else $|\sum \lambda_j \alpha_{\nu_j}| - \sum |\lambda_j| r_{\nu_j} > \rho \|\sum \lambda_j x_{\nu_j}\|$ has a solution λ; with corresponding minimax theorems and applications to unitary n-space and complex Hilbert space. The following comprehensive theorem on a continuous linear transformation A from one complex normed linear space X into another Y is proved: if Y is reflexive and if $A\{\|x\| \leqq 1\}$ is closed, then either $\|Ax - y_0\| \leqq \delta$, $\|x\| \leqq \rho$ has a solution

x or else $|g(y_0)|>\rho\|A^*g\|+\delta\|g\|$ has a solution g in Y^*, the conjugate space of Y.

The paper is concluded by an appropriate 31-item bibliography. *T. S. Motzkin* (Los Angeles, Calif.).

Lowdenslager, D. B. Duality in vector lattices. Proc. Amer. Math. Soc. 8 (1957), 476–483.

In this paper the author proves a general theorem on the duality of L_p spaces. Let L be a complete vector lattice; then the real-valued linear functional f is called o-continuous in the sense of McShane [Order-preserving maps and integration processes, Princeton, 1953; MR 15, 19] if for every net $\{x_\alpha|\alpha\in A\}$ in L with $\wedge_\beta\vee_{\alpha>\beta}x_\alpha=\vee_\beta\wedge_{\alpha>\beta}x_\alpha=x$, we have $\lim f(x_\alpha)=f(x)$. Since f is bounded (the set $\{f(x): 0\leqq x\leqq y\}$ is bounded for each y) it follows that f and the zero function have a greatest lower bound and a least upper bound in the lattice of all bounded linear functionals. The set B of all bounded linear functionals on a vector lattice is a complete lattice again. Let (L, M) be a pair of complete vector lattices and let f be a bilinear functional $\langle x, y\rangle$ on $L\times M$, such that that the functionals $y^*: x\to\langle x, y\rangle$ and $x^*: y\to\langle x, y\rangle$ are o-continuous for $x\in L$, $y\in M$ and such that the mappings $x\to x^*$ and $y\to y^*$ are 1-1 ($\langle x, y\rangle$ is termed nondegenerate). The pair (L, M) is called paired with respect to $\langle x, y\rangle$ if each vector lattice is a sublattice of the lattice of all bounded linear functionals of the other. The main theorem that is proved is the following. If (L, M) is a pair of complete vector lattices paired with respect to the nondegenerate bilinear functional $\{\langle x, y\rangle|x\in L, y\in M\}$ then every positive o-continuous linear functional on M is the supremum of a rising collection of functionals of the form x^*, $x\in L$. As an application of this theorem the author first treats the reflexivity of the real function spaces $L_p(m)$, $p>1$, where m is an arbitrary countably additive measure. It follows that L_p and L_q for $p^{-1}+q^{-1}=1$ are complete vector lattices in which order convergence implies norm convergence. L_q is a complete lattice of o-continuous linear functionals on L_p such that the natural norm and order in L_q are the natural norm and order that it has as a lattice of o-continuous linear functionals on a Banach lattice L_p. It follows that all the elements of L_q are induced by elements of L_p. Then the duality of $L_1(m)$ and $L_\infty(m)$ is treated. Here L_∞ need not be a complete vector lattice or the dual of L_1. Let m' be a measure defined on the same sets as the measure m of a measure space; then the measure space has the Radon-Nikodym property if $m'(B)\leqq Cm(B)$ for some constant C implies the existence of an element f of L_∞ such that $m'(B)=\int f\,dm$. This property is equivalent to the classical Radon-Nikodym property. The Riesz property holds if for any bounded linear functional F on $L_1(m)$ there exists an element $f\in L_\infty(m)$, such that $F(g)=\int gf\,dm$ for $g\in L_1$. Then it follows that for any measure the following conditions are equivalent: the Radon-Nikodym property holds; the Riesz property holds; L_∞ is a complete vector lattice in its natural order; and every o-continuous linear functional on L_∞ is induced by an element of L_1. As a final application of his theorem the author proves the following. A commutative *-algebra A of operators on a Hilbert space which contains the supremum of every bounded rising net in it is closed in the strongest operator topology and thus in the weak topology. *Ph. Dwinger.*

Halperin, Israel; and Luxemburg, W. A. J. Reflexivity of the length function. Proc. Amer. Math. Soc. 8 (1957), 496–499.

If λ is a length function [Ellis and Halperin, Canad. J. Math. 5 (1953), 576–592; MR 15, 439] its conjugate λ^* is defined by

$$\lambda^*(v)=\sup\left\{\int uv\,d\gamma\mid \lambda(u)\leqq 1\right\}.$$

λ is called reflexive in case $\lambda^{**}=\lambda$. The authors prove: (a) $\lambda(u)\geqq\lambda^{**}(u)\geqq\sup\{\lambda(u_e)|\ \gamma(e)<\infty\}$, where $u_e=u\cdot\chi_e$; (b) $\lambda^{**}(u)=\sup\{\lambda(u_e)|\ \gamma(e)<\infty\}$ if $\lambda^{**}(u)<\infty$ or if there are no purely infinite sets E (i.e., sets E such that $\gamma(E)=\infty$ and such that if $E\supset F$ and $\gamma(F)<\infty$ then $\gamma(F)=0$). Finally $\lambda^{**}=\lambda$ if and only if for each u either (a) $\lambda(u)=\sup\{\lambda(u_e)|\ \gamma(e)<\infty\}$ or (b) there is a purely infinite set E, depending on u, such that $\lambda(u_F)=$ for all $F\subset E$ such that $\gamma(F)>0$. Examples are given for the case $\lambda\neq\lambda^{**}$. The examples are good because they come so close to the theorems and point up the sharpness of the latter.

B. Gelbaum (Minneapolis, Minn.).

Łojasiewicz, S. Sur la valeur d'une distribution dans un point. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 239–242.

Results are sketched [for details see the paper reviewed below]. Let T be a distribution in one variable. The value $T(x_0)$ of T at x_0 is defined to be the distribution limit of $T(x_0+\lambda x)$ as λ approaches 0 (if this limit exists). Zieleźny showed that $T(x_0)$ must be a constant [same Bull. 3 (1955), 519–520; MR 17, 595; in the review, this result is attributed, apparently incorrectly, to Łojasiewicz, who was reported to have another unpublished proof]. It is now stated that $T(x_0)$ is defined if and only if $T=F^{(n)}$ for some $F(x)=C(x-x_0)^n/n!+o(|x-x_0|^n)$, and then $T(x_0)=C$; a theorem of De l'Hôpital for distributions, viz., $(T/\alpha)(x_0)=(T'/\alpha')(x_0)$, is formulated; if $T(x_0)$ exists for all x_0 and is a summable function then this latter function must coincide with the distribution $T(x)$; scalar products of two distributions and definite integrals are expressed in terms of the values of distributions at a point; and a generalization of the Denjoy integral is obtained. Finally, extensions are given for distributions in several variables, e.g., $T(x_0, y)$, the value of $T(x, y)$ at $x=x_0$, is defined to be the distribution limit of $T(x_0+\lambda x, y)$ as λ approaches 0.

I. Halperin (Kingston, Ont.).

Łojasiewicz, S. Sur la valeur et la limite d'une distribution en un point. Studia Math. 16 (1957), 1–36.

Let R be the real line and $u\to x(u)$ a sufficiently differentiable map of R into R with $x'(u)\neq 0$. For T a distribution on R we denote by $T(x(u), \psi)$ the transform of T by x, that is, $(T(x(u)), \psi)=(T, |u'|\psi)$ for any C^∞ function ψ of compact support on R, where $y\to u(y)$ is the inverse of $u\to x(u)$. We say that the distribution T defined in the neighborhood N of $x_0\in R$ has the value C at x_0 and we write $T(x_0)=C$ if the limit $\lim_{\lambda\to 0}T(x_0+\lambda x)$ exists in the topology of the space of distributions on N and is the constant C. Theorem: $T(x_0)=C$ if and only if there exists a function F continuous in the neighborhood of x_0 so that $T=F^{(n)}$ and $\lim_{x\to x_0}F(x)/(x-x_0)^n=C/n!$, or, what is the same thing, $F(x)=a_0+\cdots+C(x-x_0)^n/n!+o(|x-x_0|^n)$. Corollary: If T has a value at x_0 so does its primitive.

An interesting example is the distribution T defined by the function $f(x)=|x|^{-\alpha}\sin|x|^{-\beta}$ for $\beta>0$ and α arbitrary. Although f is not locally integrable in the neighborhood of $x=0$, f can be written in terms of derivatives (in the sense of distribution) of locally integrable function. The author shows that $T(0)=0$.

Let T, S be distributions such that $(T*S)(0)$ exists; then we can call $(T*S)(0)=(T, S)$ the scalar product of T

with S. If we choose $S=X_{ab}$, the characteristic function of the interval $[a, b]$, then we can write $(T*X_{ab})=\int_a^b Tdx$. In particular, this method allows us to extend the Denjoy integral, e.g., to the functions $|x|^{-\alpha}\sin|x|^{-\beta}$ for $\beta>0$.

There are several other similar concepts introduced in the paper. *L. Ehrenpreis* (Waltham, Mass.).

Dye, H. A.; and Phillips, R. S. Groups of positive operators. Canad. J. Math. 8 (1956), 462–486.

Let $C_0(X)$ denote the space of all real-valued continuous functions $f(x)$ vanishing at infinity of the locally compact Hausdorff space X, metrized by the norm $\|f\|=\sup_x|f(x)|$, and let $\sigma\to U_\sigma$ be a representation of the group G of homeomorphisms of X by bounded positive operators on $C_0(X)$. Then U_σ admits a factorization $U_\sigma=L_{\theta(x,\sigma)}T_\sigma$ into the flow T_σ $(T_\sigma f(x)=f(x\sigma))$ and the multiplication L_θ $(L_{\theta(x,\sigma)}f(x)=\theta(x,\sigma)f(x))$, where, for each σ, $\theta(x,\sigma)$ is a positive continuous function on X bounded away from 0 and ∞ and satisfying $\theta(x,\sigma\tau)=\theta(x\sigma,\tau)\theta(x,\sigma)$, $\theta(x,e)=1$ (e is the identity of G). The representation U_σ is equivalent to a pure flow $T_\sigma^{(1)}$ $(L_\theta U_\sigma L_\theta^{-1}=T_\sigma^{(1)})$ if and only if θ has the form $\theta(x,\sigma)=g(x)/g(x\sigma)$. Such is the case if the flow $x\to x\sigma$ is ergodic. The automorphism group of the group of all positive operators on $C_0(X)$ pertaining to a given flow is discussed, yielding the characterization of the group of flow-related automorphisms modulo inner automorphisms. In these discussions, a cohomology argument on $\theta(x,\sigma)$ is introduced as an algebraic vehicle. The canonical factorization $U_\sigma=L_\theta T_\sigma$ implies that the infinitesimal generator of a strongly continuous one-parameter group of bounded positive operators on $C_0(X)$ is, under appropriate conditions, represented as a differential operator of the first order. *K. Yosida* (Tokyo).

See also: Measure, Integration: Choquet. **Partial Differential Equations:** Séminaire Schwartz; Hörmander and Lions; Malgrange; Schwartz; Yosida. **Topological Groups:** Pettis. **Lie Groups and Algebras:** Ehrenpreis. **Convex Domains, Integral Geometry:** Goldman and Tucker. **Numerical Methods:** Schröder. **Probability:** Dynkin and Jushkevich; Dynkin; Kendall and Reuter.

Banach Spaces, Banach Algebras, Hilbert Spaces

Swerling, P. Families of transformations in the function spaces H^p. Pacific J. Math. 7 (1957), 1015–1029.

H^p is the space of all functions which are analytic in the open unit disc Δ and for which

$$\|f\|_p=\sup_{r<1}\left\{(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right\}^{1/p}<\infty.$$

The author studies semigroups of bounded linear transformations T_t of H^p into H^p, of the following type. Let $\mathcal{T}$ be an additive semigroup of complex numbers, which contains a closed ray starting at the origin, and in which every pair of points can be connected by a rectifiable path; for any $t\in\mathcal{T}$, let w_t be an analytic mapping of Δ into Δ, such that $w_0(z)=z$, $w_t(w_s(z))=w_s(w_t(z))=w_{s+t}(z)$, and $\lim(w_t(z)-z)/t$, as $t\to 0$ within $\mathcal{T}$, is an analytic function $P(z)$ defined in Δ; for each $t\in\mathcal{T}$, put $(T_tf)(z)=f(w_t(z))$. The generator A of $\{T_t\}$ is defined by $Af=\lim(T_tf-f)/t$ as $t\to 0$ within T; the domain of A consists of those f for which the limit exists in the H^p norm. It turns out that $Af(z)=P(z)f'(z)$. The main results of the paper concern (1) differentiability properties of $\{T_t\}$, (2) the structure of the family $\{T_t\}$ generated by a given operator of the form $Af(z)=P(z)f'(z)$, under certain conditions on P. The following examples are discussed: (A) $\mathcal{T}$ the closed right half plane, $w_t(z)=ze^{-t}$, $P(z)=-z$; (B) $\mathcal{T}$ the real axis, $w_t(z)=(1+az)^{-1}(z+a)$, where $a=\tanh t$, $P(z)=1-z^2$.

W. Rudin (Rochester, N.Y.).

Kendall, D. G.; and Reuter, G. E. H. Some ergodic theorems for one-parameter semigroups of operators. Philos. Trans. Roy. Soc. London. Ser. A. 249 (1956), 151–177.

By a systematic use of Eberlein's general ergodic theorem [Trans. Amer. Math. Soc. 67 (1949), 217–240; MR 12, 112], the title investigation is developed as an extension of Hille's work in his treatise [Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; MR 9, 594]. Let T_t $(t\geq 0)$ be a one-parameter semigroup of bounded linear operators on a Banach space X, strongly continuous for $t>0$, $\int_0^1\|T_tx\|dt<\infty$ on X and $\lim_{t\uparrow\infty}t^{-1}\log\|T_t\|\leq 0$. The averages of T_t considered are $\lambda J_\lambda x=\int_0^\infty\lambda\exp(-\lambda t)T_tx\,dt$ (for $\lambda\downarrow 0$), $C_\lambda^k x=k\lambda^{-k}\int_0^\lambda(\lambda-t)^{k-1}T_tx\,dt$ (for $\lambda\uparrow\infty$), $B_n^\lambda x=n^{-1}\sum_{m=1}^n(\lambda J_\lambda)^m x$ (for $n\uparrow\infty$). In particular, λJ_λ is a uniform average in the sense that $\lim_{\lambda\downarrow 0}\|\lambda J_\lambda T_t-\lambda J_\lambda\|=0$. Thus the boundedness of $\|\lambda J_\lambda\|$ for $0<\lambda<1$ implies the equivalence of strong-$\lim_{\lambda\downarrow 0}\lambda J_\lambda x=y$ with weak-$\lim_{\lambda\downarrow 0}\lambda J_\lambda x=y$. The mapping $x\to y=\Pi x$ is the (Abel) ergodic projection of the semigroup T_t. If T_t satisfies the further condition strong-$\lim_{h\downarrow 0}h^{-1}\int_0^h T_tx\,dt=x$ on X, we may define the infinitesimal generator Ω of T_t: $\Omega x=$strong-$\lim_{t\downarrow 0}t^{-1}(T_t-I)x$ for those x for which the lim exists. In this case, it is proved that $R(\Pi)=N(\Omega)$, $N(\Pi)=$ the strong closure of $R(\Omega)$, R and N denoting the range and the null space. It is proved also that $d^*=\dim N(\Omega^*)\geq d=\dim N(\Omega)$; the equality implies and is implied, at least when $d<\infty$, by the (Abel) ergodicity of T_t, i.e., by the condition that for every $x\in X$ strong-$\lim_{\lambda\downarrow 0}\lambda J_\lambda x$ exists. A subspace V of X^* defines a V-weak topology. The corresponding V-weak (Abel) ergodicity is investigated in connection with the adjoint semi-group of T_t, introduced by Phillips [Pacific J. Math. 5 (1955), 269–283; MR 17, 64]. *K. Yosida* (Tokyo).

Shimoda, Isae; Watanabe, Takeshi; and Terai, Masaaki. Notes on power series in abstract spaces. J. Gakugei Tokushima Univ. Nat. Sci. Math. 7 (1956), 67–72.

Let f be analytic at $x=0$ in E, with values in E' (E and E' are complex Banach spaces). Let the power series development at $x=0$ be $f(x)=\sum_0^\infty h_n(x)$, where h_n is a homogeneous polynomial of degree n, with modulus M_n. The radius of bound of f is defined as the least upper bound λ of numbers $r>0$ such that f is analytic and bounded when $\|x\|<r$. In a previous paper [Math. Japon. 1 (1948), 69–73; MR 10, 307] it was remarked that λ is the radius of convergence of the real series $\sum M_nt^n$. The radius of analyticity of f is the least upper bound τ of numbers $r>0$ such that f is analytic when $\|x\|<r$. It is known that $\tau^{-1}=\sup_{\|x\|=1}\limsup_{n\to\infty}\|h_n(x)\|^{1/n}$ [see the above cited review and the reference to Hille therein]. The present paper deals with estimates for λ and τ based on the quotients $\|h_n(x)\|/\|h_{n-1}(x)\|$. Some of the results are obvious in view of the known relations between the sequences $a_n^{1/n}$ and a_{n+1}/a_n, where $a_n>0$.

A. E. Taylor (Los Angeles, Calif.).

Shimoda, Isae. Notes on general analysis. VI. Singular set. J. Gakugei Tokushima Univ. Nat. Sci. Math. 7 (1956), 1–8.

This continues a previous paper by the same author [same J. 6 (1955), 5–18; MR 17, 1114]. In general, the

subject matter is the following: E_1 and E_2 are complex Banach spaces; functions f are considered which are analytic on a part of E_1, with values in E_2. It is assumed that f is analytic except perhaps on a set S which consists of one or more closed linear manifolds, and conditions are investigated that S be a removable set of singularities. In the previous paper it was shown that S is removable if it is a single subspace of defect 2 or more. This result is now generalized to the case in which each constituent subspace of S is of defect at least 2 and the subspaces do not 'cluster' at any particular constituent subspace. Also, if S is a single subspace of defect 1, then S is removable provided that to each point $x \in S$ corresponds a neighborhood $V(x)$ of x such that f is bounded on $(E_1-S) \cap V(x)$.

A. E. Taylor (Los Angeles, Calif.).

Maurin, K. Elementare Bemerkungen über kommutative $\mathfrak{C}^*$-Algebren (Beweis einer Vermutung von Dirac). Studia Math. **16** (1957), 74–79.

The author proves two of the standard theorems from the unitary equivalence theory of self-adjoint operators.

L. H. Loomis (Cambridge, Mass.).

Kramer, Vernon A. Asymptotic perturbation series. Trans. Amer. Math. Soc. **85** (1957), 88–105.

The paper studies the validity in an asymptotic sense of the classical perturbation method as applied to eigenvalue-eigenvector problems. Kato [Math. Ann. **125** (1953), 435–447; MR **14**, 990] has derived conditions for operators of the form $H_0+\iota H_1$. These do not hold for operators with higher order terms. In case $H(t)=\sum_0^\infty H_t t^t$, H_t unbounded, the author studies when the formally obtained series $\mu(t)=\sum_0^\infty t^t\mu_t$, $\phi(t)=\sum_0^\infty t^t\phi_t$ for eigenvalues and eigenvectors are asymptotically valid to Nth order. The author imposes certain semiboundedness conditions on H_t and uses Friedrich's extensions. To get around the unboundedness of H_t the inverse operator is used, which had been studied in a previous paper [Proc. Amer. Math. Soc. **7** (1956), 429–437; MR **18**, 588]. A typical theorem is (3.1): If

$$A_t=\sum_{\Sigma p_k=t}(-1)^r H_0^{-1}H_{p_1}H_0^{-1}H_{p_2}H_0^{-1}\cdots H_{p_r}H_0^{-1},$$

then with the above notation the formally derived series are

$$\mu(t)=\sum_0^{2N} t^t\mu_t+o(t^{2N}),\quad \phi(t)=\sum_0^{N} t^t\phi_t+o(t^N),$$

if $A_k\phi_l \in \mathfrak{D}(H(t))$ for $i+k<N$. The above theorem deals with a simple unperturbed eigenvalue .The "degenerate" case yields further theorems. Here the formal proofs are restricted to first order only, but splitting at higher order is discussed briefly. In the last paragraph the adiabatic transformation is used to obtain uniquely determined and "naturally distinguished" eigenvectors.

František Wolf (Berkeley, Calif.).

Širokov, F. V. Proof of a conjecture of Kaplansky. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 167–168. (Russian)

The conjecture is: if in a Banach algebra $ab-ba$ commutes with a, then $ab-ba$ is generalized nilpotent. For an independent proof by Kleinecke see the paper reviewed below. The two proofs are similar in spirit but differ in detail; the present author uses the exponential function, while Kleinecke relies on formal manipulation of commutators.

I. Kaplansky (Princeton, N.J.).

Kleinecke, David C. On operator commutators. Proc. Amer. Math. Soc. **8** (1957), 535–536.

The author gives a neat proof of the following theorem. If A and T are elements of a Banach algebra, $Q=AT-TA$ and $QA=AQ$, then $\lim_{n\to\infty}\|Q^n\|^{1/n}=0$. See also the paper reviewed above.

C. R. Putnam (Lafayette, Ind.).

Hewitt, Edwin; and Zuckerman, Herbert S. Structure theory for a class of convolution algebras. Pacific J. Math. **7** (1957), 913–941.

Let G be a totally ordered set, compact in its interval topology, and make it into a semigroup by defining $xy=\max(x, y)$ $(x, y \in G)$. The authors show that the convolution algebra $\tilde{\mathfrak{C}}(G)$ of complex Radon measures on G is a commutative Banach algebra with unit. The homomorphisms of $\tilde{\mathfrak{C}}(G)$ into the complex field are all of one of the two forms $\lambda\to\lambda(I_a^+)$, $\lambda\to\lambda(I_a^-)$, where $I_a^+=\{x: \alpha\leq x\leq a\}$, $I_a^-=\{x: \alpha\leq x<a\}$, and α is the smallest element of G. The maximal ideal space can be identified with the semigroup $\hat{G}$ of all semicharacters of G [defined by the authors in Acta Math. **93** (1955), 67–119; MR **17**, 1048]. The Gel'fand topology on $\hat{G}$ is described explicitly; the kernel-hull topology is identical with it. The algebra $\tilde{\mathfrak{C}}(G)$ is semisimple, and the Fourier transforms of Radon measures on G are precisely the continuous functions of bounded variation on $\hat{G}$, with a natural definition of the total variation.

If G is the closed interval $[0, 1]$ of the real line, $\tilde{\mathfrak{C}}(G)$ can be identified with the set of complex functions of bounded variation on G that are continuous on the right; the homomorphisms of $\tilde{\mathfrak{C}}(G)$ into the complex field are then either of the form $\varphi\to\varphi(t)$ or of the form $\varphi\to\varphi(t-0)$.

Some other applications and special results are given, and an extension of the theory to the case when G is only locally compact is outlined.

F. Smithies.

See also: Measure, Integration: Prékopa. **Approximations, Orthogonal Functions:** Paszkowski. **Trigonometric Series and Integrals:** Krabbe. **Ordinary Differential Equations:** Hebroni. **Partial Differential Equations:** Séminaire Schwartz; Huet. **Calculus of Variations:** Schäfke. **Topological Groups:** Koosis. **Topological Vector Spaces:** Phelps; Fan; Lowdenslager; Halperin and Luxemburg.

TOPOLOGY

General Topology

Weier, Josef. Über offene Euklidische Mengen. Math. Nachr. **15** (1956), 293–298.

Let U be an open connected subset of a euclidean space, and let f be any mapping of U into itself. Given any point P in U and any integer k, there is another mapping, g, of U into itself which is homotopic to f and such that the only fixed point of g is P, and such that the index of this fixed point is k.

The interest in this theorem lies in the contrast between this case, where U is open, and the case of a compact absolute neighborhood retract.

E. G. Begle.

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Rodnyanskiĭ, A. M. On continuous and differentiable mappings of open sets of Euclidean space. Mat. Sb. N.S. **42(84)** (1957), 179–196. (Russian)

Let f be a continuous map of the open connected set G in Euclidean space E^n into E^n. A point $a \in G$ is called regular if it is an isolated point of $f^{-1}(f(a))$ and if the (local) degree of f at a is not zero. Theorem 1: Suppose a is regular, U a neighborhood of a. There exists a neighborhood V of $f(a)$ such that to every connected neighborhood O_y of $f(a)$, $O_y \subset V$, one can find a connected neighborhood O_x of a, $\bar{O}_x \subset U$, with $f(O_x)=O_y$, $f(\partial O_x)=\partial O_y$ (∂=boundary). Theorem 2: If all points are regular, then f is open and the points where f is topological are open-dense in G. Theorem 3 (the Darboux property): If f has a non-singular differential at each point, then the Jacobian has constant sign in G. No further assumption is made on the partial derivatives. [That at any point f is regular with local degree ± 1 is well-known; see Alexandroff and Hopf, Topologie, Bd. I, Springer, Berlin, 1935, p. 478.] Theorem 6: Under the hypothesis of Theorem 4, f is a local homeomorphism. Some other related results are given. The proofs rest on the standard properties of local and global degree. *H. Samelson* (Ann Arbor, Mich.).

Fort, M. K., Jr. A geometric problem of Sherman Stein. Pacific J. Math. **6** (1956), 607–609.

The problem of Stein [Bull. Amer. Math. Soc. **61** (1955), 465] is as follows: Let $J \subset R_2$ be a rectifiable Jordan curve, with the property that for each rotation R there is a translation T, depending on R, such that $(TRJ) \cap J$ has a nonzero length. Must J contain the arc of a circle? Interpreting "length" to be the measure induced on J by arc length, the author constructs an example to show that J need not contain the arc of a circle. He then proves the attractive lemma that if A and B are topological arcs in R_2 and A contains an infinite number of subarcs, each of which is congruent to B, then B is either an arc of a circle or a segment of a straight line. With the help of this lemma he now establishes his own theorem that if $J \subset R_2$ is a (not necessarily rectifiable) Jordan curve, and H is an uncountable set of rotations about some one point such that for each $R \in H$ there is a translation T such that $(TRJ) \cap J$ has a nondegenerate component, then J contains an arc of a circle. There is a final observation to the effect that "uncountable" cannot be reduced to "infinite" in this theorem. *W. W. S. Claytor.*

Wagner, F. J. Notes on compactification. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 171–176, 177–181.

The author constructs all compactifications of a completely regular space X. For this purpose he studies filter-spaces over X defined as follows. Let φ be a collection of filters on X containing all filters of neighborhoods of points of X and such that every filter of φ has a base of open sets. Then φ can be topologized by taking as open sets the sets $\nu(U)$, where $\nu(U)$=the set of filters of φ containing the open set $U \subset X$. The author shows that a regular space containing a dense subset homeomorphic to X is a filter space over X. Compact filter-spaces are characterized and the Stone-Čech and one-point compactifications singled out. *M. E. Shanks* (Princeton, N.J.).

Williams, R. F. The effect of maps upon the dimension of subsets of the domain space. Proc. Amer. Math. Soc. **8** (1957), 580–583.

Considerable progress has been made recently in the study of the effect of mappings on dimension. The author is concerned with the way mappings affect the dimension of subspaces of the domain space. Typical among his results are the following. Suppose X and Y are separable metric spaces, dim $X=n$, dim $Y=m$, and $f: X \to Y$. Then (Theorem A) there is a closed subset of X of dimension at least $(n-1)/2$ whose image is of dimension at most $m/2$; (Corollary 1) if $m<2n$, then f does not raise the dimension of all closed 1-dimensional subsets of X; and (Theorem C) if X is compact, it contains a closed subset of dimension $n-1$ whose image is of dimension $\leq m-1$. Several related results are also obtained. *E. Dyer* (Chicago, Ill.).

Mansfield, M. J. On countably paracompact normal spaces. Canad. J. Math. **9** (1957), 443–449.

The author obtains some new characterizations of countably paracompact normal spaces, some of which are analogous to known characterizations of paracompact spaces. For example, in a normal space X each of the following properties is shown to be equivalent to countable paracompactness: (1) Each countable open covering of X admits a σ-closure-preserving closed refinement; (2) for each countable locally finite collection $\{A_i\}$ of subsets of X there is a countable locally finite collection $\{G_i\}$ of open subsets of X such that $A_i \subset G_i$. *E. Michael.*

Nagata, Jun-iti. Correction to the paper 'On coverings and continuous functions' (this journal, Vol. 7, 1-2, 1956). J. Inst. Polytech. Osaka City Univ. Ser. A. **8** (1957), 91.

Some minor corrections to the paper mentioned in the title [same J. **7** (1956), 29–38; MR **18**, 324]. In Lemma 2, the statement should include the requirement that $\phi(x, y)=\phi(y, x)$, and the proof must be altered near the end. Moreover, it follows from K. Morita [Proc. Japan Acad. **27** (1951), 632–636; MR **14**, 571] that, in the statement of Corollary 9, "star-refinement" may be replaced by "Δ-refinement". *E. Michael* (Seattle, Wash.).

Švarc, A. S. Proximity spaces and lattices. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 55–60. (Russian)

The author introduces the notion of a δ-lattice, which is defined as a lattice (containing the greatest and the least element) together with a binary relation $\ll$ satisfying several axioms (e.g., $x \leq y \ll z \leq t$ implies $x \ll t$; if $x \ll y$, then for some z, $x \ll z \ll y$; if $y \ll z$ whenever $x \ll z$, then $y \leq x$). If R is a proximity space satisfying the separation (normality) axiom, then the lattice of all closed $A \subset R$ is a δ-lattice if we put $A \ll B$ if and only if B is a δ-neighborhood of A (i.e., $A\delta(R-B)$). For any δ-lattice K a topology is defined: $G \subset K$ is open if and only if, for any $x \in G$, there exists $y \in G$ such that $x \ll y$, $\{z | x \leq z \leq y\} \subset G$; with this topology, K is a completely regular space. It is proved that every δ-lattice K admits of an isomorphism into the δ-lattice of all closed subsets of a suitable proximity space R (we may put $R=K$, with a suitable δ-relation in K). If K, L are δ-lattices, $K \subset L$, K dense in L (considered as a topological space), then L is called an extension of K; K is called absolutely closed if it has no proper extension (example: the δ-lattice of all closed subsets of a compact proximity space). It is shown that every δ-lattice has an (essentially unique) absolutely closed extension. *M. Katětov* (Prague).

Yarutkin, N. G. Generalized infinitesimal spaces. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 61–79. (Russian)

Various generalizations of proximity spaces are given;

the following one ("quasi-infinitesimal" spaces) is considered at length: The separation axiom (if $A\delta B$, then, for some X, $A\delta X$, $B\delta(P-X)$) is replaced by (i) $\bar{A}\delta B$ implies $A\delta B$, (ii) if $x\in P$, $x\delta A$, then there exist A_i, B_i such that $\bigcup_1^n A_i=A$, $A_i\delta B_i$, B_i are open, $x\delta B_i$. It is shown that every "quasi-infinitesimal" space has an absolutely closed extension [cf. Yu. M. Smirnov, Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 895–898; Mat. Sb. N.S. **31**(73) (1952), 543–574; MR **14**, 1107]. *M. Katětov* (Prague).

Scorza Dragoni, Giuseppe. Sulle traslazioni piane generalizzate. Abh. Math. Sem. Univ. Hamburg **21** (1957), 13–43.

The purpose of this work is to clarify the methods used by the author in his short proof of the Brouwer translation theorem [Ann. Mat. Pura Appl. (4) **39** (1955), 1–10; MR **17**, 772] and to free them from dependence on earlier literature. This is accomplished by some 68 propositions. The author hints that the short proof in question (but clarified as above) can be used for establishing Poincaré's last geometric theorem. *P. A. Smith* (New York, N.Y.).

Munkres, James. The triangulation of locally triangulable spaces. Acta Math. **97** (1957), 67–93.

Let S be a separable metric space and suppose that S is locally triangulable, in the sense that every point of S has a neighborhood which is homeomorphic to an open subset of a finite polyhedron. The author shows that if dim $S\leqq 3$, then S is triangulable (in the large). The proof uses the results, but not the methods, of papers of R. H. Bing [Ann. of Math. (2) **59** (1954), 145–158; MR **15**, 816] and the reviewer [ibid. **56** (1952), 96–114; **59** (1954), 159–170; MR **14**, 72; **15**, 889]; in effect, the theorem is reduced to the case in which S is a 3-manifold with boundary. The reduction argument, as given, is special for dimension 3. Thus even if it were known that all n-manifolds with boundary are triangulable, the triangulability of higher dimensional locally triangulable spaces would remain an unsolved problem. *E. E. Moise* (Ann Arbor, Mich.).

Barrett, Lida K. On a question concerning partitioning raised by R. H. Bing. Proc. Amer. Math. Soc. **8** (1957), 602–603.

Bing [Bull. Amer. Math. Soc. **58** (1952), 536–556; MR **14**, 192] has raised the question whether there is an n for which the following condition holds. Let M be a locally connected continuum, with distance function prescribed; let H and K be disjoint closed subsets of M; let ε be a positive number; and let R be a finite set such that for each point x of R there is an arc of diameter less than ε, containing x and intersecting both H and K. Then there are collections A_H, A_K of arcs, such that (1) $R\subset A_H\cup A_K$; (2) each element of A_H intersects H but not K; (3) each element of A_K intersects K but not H; (4) no element of A_H intersects an element of A_K; and (5) each element of $A_H\cup A_K$ is of diameter less than $n\varepsilon$. {This problem was suggested by an erroneous argument due to the reviewer, ibid. **55** (1949), 1111–1121; Proc. Amer. Math. Soc. **2** (1951), 838; MR **11**, 734; **13**, 265.}

Bing had shown [in the paper cited above] that the answer to this question is negative for $n=1$. The author shows that the answer is also negative for $n=2$.

E. E. Moise (Ann Arbor, Mich.).

Swingle, Paul M. Local properties and sums of trajectories. Portugal. Math. **15** (1956), 89–103.

It is assumed that the space S satisfies Axioms 0 and 1 of R. L. Moore [Foundations of point set theory, Amer. Math. Soc. Colloq. Publ. vol. 13, New York, 1932]. Definitions: (1) A connexe is a nondegenerate connected set. (2) Let M be a connexe, and suppose that if M is the union of two connexes A and B, then one of the sets A and B is dense in M. Then M is indecomposable. (3) Suppose that the connexe M is the union of n connexes M_i, no one of which lies in the union of the closures of the others; and suppose that the integer n is maximal with respect to this property. Then M is n-indecomposable. Local n-indecomposability at a point is also defined, as an extension of this idea; and M is said to be a local n-indecomposable net at a point x of the closure $\bar{M}$ of M if there are arbitrarily small open neighborhoods N, of x in S, such that $N\cap M$ is n-indecomposable. (4) Let M be a connexe; let $C_1, C_2, \cdots, C_n$ be connexes which are closed in M; for each i, let $E(C_i)$ be the set of all points of C_i that do not lie in the union of the sets $\bar{C}_j$ for $j\neq i$; let $W=\bigcup C_i$; and suppose that each set $E(C_i)$ is nondegenerate, and that none of the sets $E(C_i)$ lies in the closure of $M-W$. Then M has an essential content of the sets C_i. (Local essential content is defined similarly.) (5) Let M be a connexe, and let $x\in\bar{M}$. Suppose every open set U containing x contains an open set V, and a set N of power n, lying in $\bar{M}$, such that if W is a subconnexe of M, and W contains N, then $\bar{W}$ contains $V\cap M$; and suppose that n is minimal with respect to the above property. Then M is locally n-anchored at x. For further (and similar) definitions, the reader is referred to the paper. The following theorems are typical.

Theorem: There exists in the plane a connexe M, containing a point q, such that M is locally 2-indecomposable at q, but M does not contain an indecomposable connexe. Theorem: A necessary and sufficient condition that the connexe M be locally n-indecomposable at a point x of $\bar{M}$ is that that M have local essential content of connexes C_j $(1\leqq j\leqq m)$ for $m=n$ but not for $m=n+1$. Theorem: If M is a connexe, and n is an integer >1, and M' is the set of all points x of $\bar{M}$ such that M is locally n-indecomposable at x, then M' is nowhere dense in M. Theorem: Let M be a locally compact connexe, in a completely separable space. If M is locally n-indecomposable at a point x of $\bar{M}$, where $n>1$, then M is locally n'-anchored at x, for some $n'\leqq n$. Theorem: A necessary and sufficient condition that the (compact) continuum M be locally n-indecomposable at a point x of M is that M be both a local n-indecomposable net at x and locally n'-irreducible at x. *E. E. Moise* (Ann Arbor, Mich.).

Lintz, Rubens G. Sur le théorème de Jordan dans un espace T_1. Ann. Mat. Pura Appl. (4) **43** (1957), 357–370.

The Jordan proposition may be formulated for the plane as follows: The complement of any closed curve consists of two parts (1) each open, (2) each having the curve as frontier, and (3) each connected. Starting from a definition of open curve as a kind of irreducible continuum, and of closed curve in terms of open curves, the author postulates (a) a local arc-wise connectedness, (b) the existence of suitable closed curves, (c) the Jordan proposition weakened by the omission of (3). The first main result is that the Jordan proposition holds in its complete form. The second result is that the points of the curve are, in a sense defined in the paper, accessible from the complement. *R. Arens* (Los Angeles, Calif.).

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Fadell, Edward. On spaces without isolated noncut points. Proc. Amer. Math. Soc. 8 (1957), 362–365.

Let X be a topological space and let $f:X\to X$ be a mapping. If $f(x)\neq x$, then the point x is free. If for every $x\in X$ there is an $x'\neq x$ such that x' does not separate x from $f(x')$ in X, then f is a ϕ-mapping. Hereafter, it is assumed that X is a Peano continuum. A tail of X is an arc $A=ab$ such that $A-a$ is open. Ω denotes the function space X^X; Ω_0 is the space of mappings of X onto X; Φ is the set of all ϕ-mappings; Φ' is the set of mappings for which at least one non-cut-point is free; and Φ'' is the set of mappings for which infinitely many non-cut-points are free. (By convention, the latter three classes contain the identity mapping.) Theorem: X has no tail if and only if (1) $\Phi'=\Phi''$ or (2) $\Omega=\Phi\cup\Omega_0$. Theorem: The non-cut-points of X are dense in X if and only if (1) $\Phi=\Phi'$ or (2) $\Omega=\Phi$. Theorem: Let X be an absolute neighborhood retract. If (1) X has no tail, then (2) for every mapping $f: X\to X$ there is a mapping g, of X onto X, such that g is homotopic to f under a homotopy in which the fixed points, at every stage, are precisely the fixed points of f. And conversely, (2) implies (1). *E. E. Moise.*

Mamuzić, Z. Structures topologiques définies sur un ensemble E par des structures sur un ensemble M et par application $f(E\times E)\subset M$. Univ. Beograd. Zb. Mašin. Fak. 1954–55, 1–18. (Serbo-Croatian. French summary)

Mamuzić, Zlatko. Opérateurs $\mathfrak{E}[M]$, $\mathfrak{D}[M]$ et structures topologiques sur un ensemble E. Bull. Soc. Math. Phys. Serbie 7 (1955), 39–72; errata 136. (Serbo-Croatian. French summary)

Mamuzić, Zlatko. Structures topologiques [uniformes] diverses définies sur un ensemble E par application $f(E\times E)\subset M$, dans un ensemble M ordonné [quelconque]. Bull. Soc. Math. Phys. Serbie 7 (1955), 185–216 (1956). (Serbo-Croatian. French summary)

Let E and M be arbitrary non-void sets, and let f be any single-valued mapping of the Cartesian product $E\times E$ into M. Let $\mathfrak{F}$ be any non-void family of subsets X of M such that $X\supset\{f(a,a): a\in E\}$ for all $X\in\mathfrak{F}$. For every $a\in E$ and $X\in\mathfrak{F}$, let $W_X(a)=\{b: b\in E, f(a,b)\in X\}$. This defines a family of neighborhoods of a, and the resulting space $E(M_f)$ is a neighborhood space in the sense of Fréchet. This procedure is of course reminiscent of the definition of a uniform structure on E. By using the sets $f^{-1}(X)$, one sees at once that this procedure is in fact the imposition of a generalized uniform structure on E.

Suppose next that a family of subsets $\mathfrak{F}_a$ of M is defined for each $a\in E$, with the property that $f(a,a)\in X_{\lambda_a}$ for all $X_{\lambda_a}\in\mathfrak{F}_a$. Then the sets $W_{\lambda_a}(a)=\{b: b\in E, f(a,b)\in X_{\lambda_a}\}$ may also be taken as neighborhoods of a. The author proves a number of theorems about the topologies on E produced in these two ways. We mention some of these for the topology $E(M_f)$. The closure operator $^-$ always has the property that $(\overline{\mathfrak{F}_1\cup\mathfrak{F}_2})\subset\overline{\mathfrak{F}_1}\cup\overline{\mathfrak{F}_2}$. The property $\overline{\overline{\mathfrak{F}}}=\overline{\mathfrak{F}}$ holds if and only if, for every triple of points a, b, $c\in E$ and every $X\in\mathfrak{F}$, there exist Y, $Z\in\mathfrak{F}$ such that $f(a,b)\in Y$ and $f(b,c)\in Z$ imply $f(a,c)\in X$. Necessary and sufficient conditions on f and $\mathfrak{F}$ for the validity of the T_1, T_2 and T_3 separation axioms are given, as well as a sufficient condition for the complete normality of $E(M_f)$.

The author next takes up the case in which M is a topological space and the sets X_{λ_a} defining the neighborhoods $W_{\lambda_a}(a)$ are neighborhoods of the point $f(a,a)$. The validity of various separation axioms on, and uniformizability of, E are examined in great detail, under various assumptions about M.

The final section is concerned with various analogues of uniform structures and metrics on E, and how they can be realized in terms of the mappings f defined above. *E. Hewitt* (Seattle, Wash.).

See also: Functions of Complex Variables: Maude. **Geometries, Euclidean and Other:** Guggenheimer.

Algebraic Topology

★ **Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. i+234 pp. (polycopiées). 1200 francs.

Dieses Seminar ist den von Eilenberg u. MacLane [Ann. of Math. (2) **46** (1945), 480–509; MR **7**, 137] zuerst betrachteten Räumen $\mathfrak{K}(\Pi, n)$ — das sind Räume mit Homotopiegruppen $\pi_n=\Pi$, $\pi_m=0$ $(m\neq n)$ — und einigen ihrer zahlreichen Anwendungen gewidmet. Im Protokoll 1 gibt Serre eine Zusammenstellung der wichtigsten topologischen Eigenschaften dieser Räume: ihre Homologiegruppen [vgl. Eilenberg u. MacLane, loc.cit. und ibid. **58** (1953), 55–106; **60** (1954), 49–139, 513–557; MR **15**, 54; **16**, 391, 392], Pontrjagin-Produkt in der Homologie, $\mathfrak{K}(\Pi, n)$ als Räume geschlossener Wege, Serresche Faserungen [vgl. Serre ibid. **54** (1951), 425–505; MR **13**, 574], $\mathfrak{K}(\Pi, n)$ und Kohomologie-Operationen [vgl. z.B. Serre, Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643], $\mathfrak{K}(\Pi, n)$ und Postnikovsche Systeme [Postnikov, Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 359–362, 789–791; MR **13**, 374, 375; Mizuno, J. Inst. Polytech. Osaka City Univ. Ser. A. **5** (1954), 41–51; MR **16**, 506], Anwendung der Räume $\mathfrak{K}(\Pi, n)$ in der Obstruktionstheorie [Eilenberg u. MacLane, Ann. of Math. (2) **60** (1954), 513–557; MR **16**, 392] und zur Berechnung von Homotopiegruppen [vgl. z.B. Cartan u. Serre, C. R. Acad. Sci. Paris **234** (1952), 288–290, 393–395; MR **13**, 675]. In den folgenden Protokollen werden einige dieser Punkte bzw. ihre algebraischen Analoga ausführlich behandelt, nämlich: die Cartansche Methode der „Konstruktionen" [Cartan, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 467–471; MR **16**, 390] als algebraisches Analogen zu den Serreschen Faserungen, Kohomologie-Operationen, Postnikovsche Systeme und einiges weitere. Cartan gibt in den Protokollen 2 bis 11 eine ausführliche Darstellung seiner Methode der „Konstruktionen" für *DGA*-Algebran, die die Eilenberg-MacLanesche Strichkonstruktion [bar construction; Eilenberg u. MacLane, Ann. of Math. (2) **58** (1953), 55–106; MR **15**, 54] als Spezialfall umfaßt, wobei als zusätzliches Element eine vermittelnde azyklische *DGA*-Algebra M hinzutritt [vgl. Cartan loc.cit.], die das algebraische Analogon zu dem in sich zusammenziehbaren Faserraum der Wege in $\mathfrak{K}(\Pi, n)$ mit festem Anfangspunkt, der in den Serreschen Untersuchungen [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574] auftritt, darstellt. Auf die zahlreichen Einzelheiten der Cartanschen Untersuchungen (z.B. auch bezüglich der Eilenberg-MacLaneschen Einhängungsoperation, den Bocksteinschen Operationen u.a.) kann hier nicht im einzelnen eingegangen werden. Es genüge anzuführen, daß die Untersuchungen mit einer expliziten Bestimmung der Homologiealgebren $H_*(\Pi, n; Z_p)$ und der Kohomologiealgebren $H^*(\Pi, n; Z_p)$ (Z_p=Restklas-

senkörper mod p) sowohl für den Fall einer ungeraden Primzahl p als auch für den Fall $p=2$ [vgl. hierzu auch Serre, Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643] und der ganzzahligen Homologiealgebren $H_*(\Pi, n; Z)$ abschliessen [vgl. hierzu auch die Voranzeige von Cartan, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 704–707; MR **16**, 390]. Daß die Cartanschen Konstruktionen wirklich die Homologie der Räume $\mathfrak{K}(\Pi, n)$ liefern, folgt aus dem Hauptsatz von Eilenberg u. MacLane [Ann. of Math. (2) **58** (1953), 55–106; MR **15**, 54] über den Zusammenhang zwischen Strichkonstruktion und W-Konstruktion. In den Protokollen 12 und 13 beweist Moore diesen Satz, durch den die Strichkonstruktion und damit auch die Cartanschen Konstruktionen in Beziehung gesetzt werden zu den Eilenberg-MacLaneschen Komplexen $K(\Pi, n)$, die minimale Unterkomplexe der singulären Komplexe der Räume $\mathfrak{K}(\Pi, n)$ sind. — In den Protokollen 14 bis 17 behandeln Cartan und Thom Kohomologie-Operationen [vgl. z.B. Serre, Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643; und Eilenberg u. MacLane, Ann. of Math. (2) **60** (1954), 513–557; MR **16**, 392] und entsprechende Homologie-Operationen. Und zwar wird die Theorie entwickelt für vollständige halbsimpliziale Komplexe. Insbesondere werden die additiven Kohomologie-Operationen eingehend untersucht und für den Fall $H^n(X; \Pi) \to H^q(X; G)$ mit $pG=0$ (p Primzahl) vollständig bestimmt. Sie lassen sich in gewissem Sinne auf iterierte Steenrod-Operationen und (verallgemeinerte) Bockstein-Operationen zurückführen. Die Steenrod-Operationen selbst werden durch Eigenschaften gekennzeichnet und ihre Existenz ohne Zurückgehen auf die explizite Steenrodsche Definition bewiesen. Weiter gibt Cartan einen Beweis seiner Produktformel für Steenrod-Operationen [C. R. Acad. Sci. Paris **230** (1950), 425–427; MR **12**, 42] im Rahmen der Sätze der vorigen Protokolle, und Thom beweist einige Sätze über reelle Kohomologie, u.a., daß der reelle Kohomologiering eines in Sphären ungerader Dimension gefaserten Raumes, falls er orientierbar ist, nur von der charakteristischen Klasse des Raumes abhängt. — In den Protokollen 18, 19, 21 betrachtet Moore Monoid-Komplexe. Das sind vollständige halbsimpliziale Komplexe, deren Simplexe in jeder Dimension ein Monoid (Menge mit assoziativer Verknüpfung) mit Einselement bilden. Für gewisse dieser Komplexe Γ werden Homotopiegruppen definiert, die mit den gewöhnlichen Homotopiegruppen übereinstimmen, falls Γ der singuläre Komplex des Raumes der geschlossenen Wege eines Raumes ist. Weiter werden minimale Monoidkomplexe definiert. Zu jeder abelschen Gruppe Π und ganzen Zahl n existiert genau ein minimaler Gruppenkomplex Γ mit $\pi_n(\Gamma)=\Pi$, $\pi_m(\Gamma)=0$ ($m\neq n$), nämlich der Gruppenkomplex $\Gamma(\Pi, n)$, von dem $K(\Pi, n)$ der zugehörige Gruppenring-Komplex ist. Jeder minimale abelsche Gruppenkomplex Γ ist ein Produkt $\mathbf{X}_{n=0}^{\infty} \Gamma(\pi_n(\Gamma), n)$. Weiter entwickelt Moore eine Theorie der Postnikovschen Systeme für Gruppenkomplexe, die analog ist zur Postnikovschen Methode für beliebige singuläre Komplexe [vgl. Postnikov, Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 359–362, 789–791; MR **13**, 374, 375]. Insbesondere wird die Bestimmung minimaler Gruppenkomplexe unter Zuhilfenahme von Postnikovschen Systemen untersucht. — In Protokoll 20 gibt Serre einen Beweis des Satzes von Hilton [J. London Math. Soc. **30** (1955), 154–172; MR **16**, 847] über die Homotopiegruppen eines Sphären-"Buketts" mit Anwendungen auf Homotopie-Operationen von mehreren Variablen, Jacobi-Identität für das Whitehead-Produkt, Verallgemeinerungen der Hopfschen Invarianten. — Auch in Protokoll 22 (Moore) werden Verallgemeinerungen der Hopfschen Invarianten betrachtet, ferner die Freudenthalsche Einhängung, Sätze von Samelson u. Bott, G. Whitehead [Ann. of Math. (2) **57** (1953), 209–228; MR **14**, 1110] und James [ibid. **62** (1955), 170–197; **63** (1956), 191–247, 407–429; **65** (1957), 74–107; MR **17**, 396, 1117; **18**, 58, 662].

E. Burger (Zbl. **67** (1957), 156).

★ **Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie. 2ème éd.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. iii+234 pp. (polycopiées). 1200 francs.

Hilton, P. J. On Borsuk dependence and duality. Bull. Soc. Math. Belg. **7** (1955), 143–155.

The present paper is an extension of the discussion of dependence of mappings [Fund. Math. **43** (1956), 358–386; MR **18**, 814] in which the author generalized Borsuk's original notion [Bull. Acad. Polon. Sci. Cl. III **3** (1955), 81–85, 251–254; MR **16**, 946; **17**, 652]. The main additions are the introduction of i-homotopic and p-homotopic maps of modules, dependence of such maps (with theorems analogous to those for spaces), and the dualization of the known results for spaces.

Borsuk's dependence as generalized by the author is now called i-dependence, and a dual concept of p-dependence is defined. {This should not be confused with the (polyhedral-) dependence of Hilton's earlier paper.} A map of spaces $g: Z \to X$ is p-dependent on a map $f: Y \to X$ (p for projection, i for injection) if the existence of a lifting of f into any counterimage X^1 of X implies the existence of a lifting of g into X^1. Corresponding to Borsuk's theorem, the following is proved: g p-depends on f if and only if there exists $s: Z \to Y$ such that g is homotopic to fs. Attention is called to the duality of mapping cylinder and mapping fibre space, and of homotopy extension property and covering homotopy property.

The duality is motivated by the obvious duality in the analogous situation for modules. Two maps $\phi: A \to B$, $\psi: A \to B$, of Λ-modules are called i-homotopic ($\simeq_i$) if $\phi - \psi$ is i-nullhomotopic, i.e., may be extended to every supermodule of A, or equivalently to some injective module containing A. $\psi: A \to C$ is i-dependent on $\phi: A \to B$ if the existence of an extension of ϕ to $A' \supset A$ implies the existence of an extension of ψ to A'. As for topological spaces, it is proved that: ψ i-depends on ϕ if and only if there exists $\sigma: B \to C$ such that $\psi \simeq_i \sigma\phi$. The dual notions of p-homotopy ($\simeq_p$) (being p-nullhomotopic is equivalent to being liftable into some projective counterimage of A) and p-dependence for maps of modules $\psi: C \to A$, $\phi: B \to A$ are defined completely analogously to the topological case and dually to the i-case for modules. The corresponding theorem is: ψ p-depends on ϕ if and only if there exists $\sigma: C \to B$ such that $\psi \simeq_p \phi\sigma$. The proofs involve respectively the module analogue of the mapping cylinder and the mapping fibre space.

For spaces, it is shown that p-dependence generalizes Thom's notion of subordination of cohomology classes [see the work reviewed second above]. If $\beta \in H^r(X; B)$, $\alpha \in H^r(X; A)$, let $g: X \to K(A; r)$ be such that $g^*(\varepsilon) = \alpha$, where ε is the fundamental class of $H^r(K(A; r); A)$. Let π_α be the projection of the fibre space over X induced by g from the path space over $K(A; r)$; define π_β analogously; then β is subordinate to α if and only if π_α p-depends on π_β.

Finally the n-dimensional version of the theory is

developed for spaces with Borsuk's dimensional restriction replaced by a homotopy invariant: homology dimension n ($H_n(X)$ free abelian; $H_r(X)=0$, $r<n$).

J. Stasheff.

Kan, Daniel M. Abstract homotopy. III. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 419–421.

[For parts I–II see same Proc. **41** (1955), 1092–1096; **42** (1956), 255–258; MR **18**, 142.] Let $\mathscr{S}$ denote the category of semi-simplicial complexes, and semi-simplicial maps. If K is a complex in $\mathscr{S}$, and σ is an n-simplex of K, let $\partial_i\sigma$ denote the ith face of σ for $i=0, \cdots, n$. The complex K is said to satisfy the extension condition if whenever $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n$ are $(n-1)$-simplexes of K such that $\partial_i\sigma_j=\partial_{j-1}\sigma_i$ for $i<j$, $i, j\neq k$, there exists an n-simplex σ in K such that $\partial_i\sigma=\sigma_i$ for $i\neq k$. This corresponds to the geometric idea that the union of all faces but one of dimension $n-1$ of an n-simplex is a retract of the n-simplex itself. Denote by $\mathscr{S}_E$ the subcategory of $\mathscr{S}$ whose objects are complexes satisfying the extension condition and whose maps are semi-simplicial maps between these objects. The main purpose of this note of the author is to define a functor Ex^∞: $\mathscr{S}\to\mathscr{S}_E$, and to define a natural transformation of functors e^∞: $I\to\mathrm{Ex}^\infty$, where I denotes the identity functor on $\mathscr{S}$, satisfying the following conditions: (i) for any complex K in $\mathscr{S}$, $e^\infty_{K*}: H_*(K)\simeq H_*(\mathrm{Ex}^\infty(K))$; (ii) if $f: K\to L$ is a fibre map, then $\mathrm{Ex}^\infty f: \mathrm{Ex}^\infty K\to\mathrm{Ex}^\infty L$ is a fibre map; (iii) if $K\in\mathscr{S}_E$, then $e^\infty_K: K\to\mathrm{Ex}^\infty K$ is a homotopy equivalence; and (iv) if $f, g: K\to L$ are homotopic maps in $\mathscr{S}$, then $\mathrm{Ex}^\infty f$ and $\mathrm{Ex}^\infty g$ are homotopic maps. The existence of the functor Ex^∞ enables the author to apply all the notions of ordinary homotopy theory to the category $\mathscr{S}$, for the notions of classical homotopy theory are all valid in the category $\mathscr{S}_E$.

Further suppose $|K|$ denotes the geometric realization of a complex in $\mathscr{S}$ as defined by J. W. Milnor [Ann. of Math. (2) **65** (1957), 357–362; MR **18**, 815], and for any space X, $S(X)$ denotes the singular complex of X; then (i) $S|K|$ and $\mathrm{Ex}^\infty K$ have the same homotopy type, and (ii) $S|e^\infty_K|: S|K|\to S|\mathrm{Ex}^\infty K|$ is a homotopy equivalence.

The paper contains various other useful theorems, and an outline of the construction of Ex^∞ which, although brief, should enable anyone sufficiently familiar with the theory of semi-simplicial complexes to reconstruct most of the author's proofs. *J. C. Moore* (Princeton, N.J.).

Kan, Daniel M. Abstract homotopy. IV. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 542–544.

The aim of the author in this paper is to define the homotopy groups of a simplicial complex by using only its simplicial structure. Every simplicial complex gives rise to a semi-simplicial complex, and the author proceeds to carry out his program by defining the homotopy groups of a semi-simplicial complex in a manner which depends only on the semi-simplicial structure of the complex.

A group complex G is a semi-simplicial complex such that (i) the set of n-simplexes G_n is a group, and (ii) $\partial_i: G_n\to G_{n-1}$ and $s_i: G_{n-1}\to G_n$ are homomorphisms, where for any simplex x, $\partial_i x$ is the ith face of x, and $s_i x$ is the ith degeneracy of x. For any group complex G, let $\bar{G}_n$ be $\bigcap_{i=0}^{n-1}$ (kernel $\partial_i: G_n\to G_{n-1}$). Then $x\in G_n$ implies $\partial_n x\in\bar{G}_{n-1}$ and in fact $\partial_n x\in$ kernel $\partial_{n-1}: \bar{G}_{n-1}\to\bar{G}_{n-2}$. Further, image $\partial_n: \bar{G}_n\to\bar{G}_{n-1}$ is a normal subgroup of kernel $\partial_{n-1}: \bar{G}_{n-1}\to\bar{G}_{n-2}$. Now, $\bar{G}=\sum\bar{G}_n$ is not necessarily an abelian chain complex, and $H_n(\bar{G})=\pi_n(G)$. In other words, the homotopy groups of a group complex G are just the homology groups of the chain complex $\bar{G}$. These facts about group complexes were proved by the reviewer [see the work reviewed fourth above, Exposées 19 and 20].

Now let K be a semi-simplicial complex, and K_n its set of n-simplexes. Let $\phi\in K_0$ be a fixed zero-simplex, and let $\phi^n=s_{n-1}\cdots s_0\phi$. If $\tau\in K_{n+1}$, let $\partial^{n+1}\tau$ denote its last vertex, i.e., $\partial^{n+1}\tau=\partial_0\cdots\partial_n\tau$. An n-path of K ending at σ is a sequence $(\sigma_1, \cdots, \sigma_{2l})$ of $(n+1)$-simplexes of K such that $\partial_{n+1}\sigma_1=\phi^n$, $\partial^{n+1}\sigma_{2j-1}=\partial^{n+1}\sigma_{2j}$ for $1\leq j\leq l$, $\partial_{n+1}\sigma_{2l}=\sigma$, and $\partial_{n+1}\sigma_{2j}=\partial_{n+1}\sigma_{2j+1}$ for $1\leq j<l$. It is called a reduced n-path if $\sigma_i\neq\sigma_{i+1}$ for all i. An n-path ending at ϕ^n is called an n-loop. A tree of K is a connected subcomplex $T\subset K$ which contains no reduced loops, and a maximal tree is a tree with $T_0=K_0$.

Let K be a connected semi-simplicial complex with base point $\phi\in K_0$, and T a maximal tree in K. Define a group complex G as follows: (i) G_n is the free (non-abelian) group with generators $\bar{\sigma}$ such that $\sigma\in K_{n+1}$, modulo the relations $\bar{\sigma}=1$ if $\sigma\in T_{n+1}$ or $\sigma=s_n\tau$ for some $\tau\in K_n$. (ii) ∂_i: $G_n\to G_{n-1}$ is defined by $\partial_i\bar{\sigma}=\overline{\partial_i\sigma}$ for $i<n$, and $\partial_n\bar{\sigma}=\overline{\partial_n\sigma}(\overline{\partial_{n+1}\sigma})^{-1}$. (iii) s_i: $G_{n-1}\to G_n$ is defined by $s_i\bar{\sigma}=\overline{s_i\sigma}$. Now G is a group complex, and each G_n is a free non-abelian group.

For any semi-simplicial complex K, let $|K|$ denote the geometric realization of K, and for any topological space X let SX denote the singular complex of X. We are now in a position to state the first main theorem of the author. Theorem: If K is a connected semi-simplicial complex and G is defined as above, then there is a natural isomorphism $g: \pi_n(S|K|)\approx H_{n-1}(\bar{G})$. Essentially this means that G may be viewed as a loop complex for K, i.e., G bears the same relation to K as does the space $\Omega(X, x_0)$, of loops based at x_0, to the topological space X with base point x_0.

Now let A be the group complex obtained by abelianizing G, i.e., $A_n=G_n/[G_n, G_n]$. We may now state the second main theorem of the author. Theorem: There is a natural isomorphism a: $H_n(S|K|)\approx H_{n-1}(\tilde{A})$, and a commutative diagram

$$\begin{array}{ccc} \pi_n(S|K|) & \xrightarrow{g} & H_{n-1}(\bar{G}) \\ \downarrow & & \downarrow \\ H_n(S|K|) & \xrightarrow{a} & H_{n-1}(\tilde{A}) \end{array}$$

where the vertical homorphisms are the natural ones.

J. C. Moore (Princeton, N.J.).

Shih, Weishu. Sur la condition d'extension de Kan pour les complexes semi-simpliciaux. C. R. Acad. Sci. Paris **244** (1957), 1131–1132.

A proof that the Kan condition for a semi-simplicial complex K — viz., that any simplicial map into K of the subcomplex "boundary minus one face" of a simplex extends to the simplex [see the review second above] — holds in full if it holds when the exceptional face is either the 0th or the last. The argument does not involve the degeneracy operators of K. *J. A. Zilber.*

Peterson, Franklin P. Some non-embedding problems. Bol. Soc. Mat. Mexicana (2) **2** (1957), 9–15.

The author studies the question of whether a space X can be of the same homotopy type as a subpolyhedron Y of S^n. If this is the case, Y has an n-dual Y^* in the sense of Spanier and Whitehead [Mathematika **2** (1955), 56–80; MR **17**, 653]. Then the Alexander duality gives an isomorphism of the cohomology of X onto the homology of Y^* and converts stable cohomology operations of X into

stable homology operations of Y^*. Using Pontrjagin duality between homology and cohomology of Y^* it is seen that stable cohomology operations of X correspond to dual stable cohomology operations of Y^*. Explicit calculation of these dual operations for the case of Steenrod reduced powers imposes some necessary conditions on the cohomology operations in X.

This technique is applied to prove that certain spaces are not of the same homotopy type as subpolyhedra of S^n and, hence, cannot be imbedded in S^n. For example, it is shown that, if 2^s is the highest power of 2 such that $2^s \leq r$, then if $n < 2^{s+1}$ the real projective space P^r is not of the same homotopy type as a subpolyhedron of S^n.

E. H. Spanier (Chicago, Ill.).

Cohen, D. E. Products and carrier theory. Proc. London Math. Soc. (3) **7** (1957), 219–248.

This paper starts with a study of some of the properties of the join of two topological spaces. The material developed in the first section is used to define Whitehead products in the carrier theory of Spanier and Whitehead [Algebraic geometry and topology, Princeton, 1957, pp. 330–360; MR **18**, 919]. It is shown that the Whitehead products in carrier theory have all the usual properties including anti-commutativity and bilinearity, and that they satisfy the appropriate Jacobi identity. The composition operation and the Hopf invariant are also discussed.

J. C. Moore (Princeton, N.J.).

van de Ven, A. J. H. M. An interpretation of the formulae of Kundert concerning higher obstructions. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 196–200.

The author presents a simplified version of certain results of Kundert on secondary obstructions to cross-sections in fiber bundles with projective spaces as fibers [Ann. of Math. (2) **54** (1951), 215–246; MR **13**, 374]. In particular, he gives a proof of Kundert's "1. Hauptformel".

J.-P. Meyer (Baltimore, Md.).

Chern, S. S.; Hirzebruch, F.; and Serre, J.-P. On the index of a fibered manifold. Proc. Amer. Math. Soc. **8** (1957), 587–596.

The index of a real quadratic form $F(x, x)$ is defined as $\tau = p - q$ where, with respect to a suitable base, $F(x, x) = \sum_1^p x_i^2 - \sum_{p+1}^{p+q} x_i^2$. If M is a compact oriented manifold of dimension $4k$ the cup product defines on $H^{2k}(M)$ a real quadratic form, and the index of this is $\tau(M)$. If $\dim M \not\equiv 0 \bmod 4$ we define $\tau(M) = 0$.

The authors prove the following theorem. Let $E \to B$ be a fibre bundle with fibre F such that (i) E, B, F are compact connected oriented manifolds (the orientations being compatible with one another), and (ii) $\pi_1(B)$ acts trivially on $H^*(F)$. Then $\tau(E) = \tau(B)\tau(F)$. The proof uses the cohomology spectral sequence $\{E_r\}$ of the fibre bundle. First it is shown that each E_r is a "Poincaré ring" (roughly, a ring having the properties of $H^*(M)$, where M is a compact connected manifold), and so the index $\tau(E_r)$ may be defined. Then there are three stages in the proof: (1) $\tau(E_2) = \tau(B)\tau(F)$, (2) $\tau(E_r) = \tau(E_{r+1})$, (3) $\tau(E_\infty) = \tau(E)$. Each of these expresses a purely algebraic invariance property of the index of a quadratic form and is easily verified.

M. F. Atiyah (Cambridge, England).

Papakyriakopoulos, C. D. On solid tori. Proc. London Math. Soc. (3) **7** (1957), 281–299.

By an n-manifold the author means a triangulated n-manifold, possibly with boundary, and not necessarily compact. A closed manifold is a compact manifold without boundary. A loop L is a (semi-linear) closed path; and a loop without multiple points is simple. $L \simeq L_0$ means that L is homotopic to L_0.

Theorem. Let M be a three-manifold with boundary N. Let L be a loop belonging to an open subset U of an orientable component N' of N, such that $L \simeq 0$ on M and $L \not\simeq 0$ on N. Then there is a simple loop L_0 in U, such that $L_0 \simeq 0$ on M and $L_0 \not\simeq 0$ on N.

Theorem. Let M, N and N' be as in the preceding theorem, and suppose that N' is closed. Then there is a finite set $W_1, W_2, \cdots, W_n$ of simple loops on N', with common base-point x_0, such that (1) each W_i is $\simeq 0$ on M, but not on N', (2) $W_i \cap W_j = x_0$ for $i \neq j$, and (3) if L is a loop on N', based at x_0, and $L \simeq 0$ on M, then L is homotopic on N' to a product of the form $\Pi_{i=1}^m C_j W_{i(j)}^{\epsilon_j} C_j^{-1}$, where each C_j is a loop on N' based at x_0, and each ϵ_i is $= \pm 1$.

Theorem. Let M be a three-manifold with boundary N. Suppose that M is compact, and that $\pi_1(M, N) = 1$ (which implies that N is connected). Let h be the genus of N. Then there is a canonical set of simple closed polygons $A_1, B_1, \cdots, A_h, B_h$ on N such that each B_i is $\simeq 0$ on M and $\bigcup A_i$ is a deformation retract of M.

Finally, the author proves the following, which is the solution of a problem of Bing [Summary of Lectures and Seminars, Summer Institute of Set Theoretic Topology, Madison, Wisc., Amer. Math. Soc., 1955, p. 57, problem 12].

Theorem. Let P be a plane in three-space E^3, and let H_1, H_2 be the open half-spaces which are the components of $E^3 - P$. Let $L_1, L_2, \cdots, L_n$ be lines perpendicular to P. Let L be a loop on H_1, such that $L \not\simeq 0$ on $E^3 - \bigcup L_i$. For each i from 1 to n, let B_i be a broken line in H_2, joining the end-points of an interval $I_i \subset L_i$, let $L_i' = (L_i - I_i) \cup B_i$, and suppose that (1) each L_i' is an (infinite) broken line (without multiple points) and (2) the sets L_i' are disjoint. Then $L \not\simeq 0$ on $E^3 - \bigcup L_i'$.

E. E. Moise.

★**Reeb, Georges. Les espaces localement numériques non séparés et leurs applications à un problème classique.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 12 pp.

The author calls a space "locally numerical" (L.N.) of dimension n if it is locally homeomorphic with Euclidean n-space, but does not necessarily satisfy the Hausdorff separation axiom. The definition is motivated by the fact that certain quotient spaces of manifolds are locally Euclidean, but not Hausdorff. A number of properties of L.N. spaces are studied; the notion of differentiability is introduced. For laminated structures in the plane these concepts lead to a simple proof of Kaplan's theorem [Duke Math. J. **7** (1940), 154–185; MR **2**, 322] on the existence of a continuous function whose level sets are the leaves.

H. Samelson (Ann Arbor, Mich.).

Maxwell, Charles N. Fixed points of symmetric product mappings. Proc. Amer. Math. Soc. **8** (1957), 808–815.

Let X be a polyhedron, X^n its nth cartesian product, and G a group of permutations on n letters. Then G can be considered, in an obvious way, as a group of homeomorphisms on X^n. The orbit space is denoted by X^n/G. For any map f of X into X^n/G, a Lefschetz number, $L(f)$, is defined, This is done by constructing a particular simplicial subdivision of X^n/G from a given subdivision

of X, and a particular chain mapping μ from this subdivision of X^n/G to X. It is shown that the induced mapping μ_* of the homology groups is a homotopy invariant. Now, for any mapping f of X into X^n/G, let f' be a suitably close simplicial approximation to f, and define $L(f)$ to be $L(\mu_* f_*')$.

The main theorem states that if $L(f)\neq 0$, then f has a fixed point, x, in the sense that $f(x)$ has x for one of its coordinates. Applications are given to spaces with a finite group of operators. For example, let Y be a polyhedron and G a finite group of operators on Y. Let π be the natural mapping of Y onto Y/G. It is shown that if either Y or Y/G is acyclic, any mapping f of Y into Y/G has a coincidence with π. *E. G. Begle.*

Guggenheimer, H. Complexes avec automorphismes non-admissibles. Bull. Res. Council Israel. Sect. A. **6** (1957), 83–93.

On an almost-complex manifold there are two differentials d, $\bar{d}$ operating on the group of exterior differential forms, and the integrability condition is $d\bar{d}+\bar{d}d=0$. Generalizing this situation, the author considers graded vector spaces, over a valued field K, admitting two differential operators satisfying certain commutativity conditions.

Given a chain complex C_∂, where each C_r is a vector space over K, let C_δ^* be the dual cochain complex; suppose further that there is a (1, 1) semilinear map $S_r: C_r \to C_r^*$ such that (i) $\|x\|=\sqrt{|\langle x, Sx\rangle|}$, $x\in C_r$, is a norm on C_r, (ii) $\langle x, Sy\rangle=\langle y, Sx\rangle$, and (iii) the image of S is stable under δ. Then a symmetric non-singular scalar product $(x, y)=\langle x, Sy\rangle$ is defined and C_d is a cochain complex, where $d=S^{-1}\delta S$. {Reviewer's note: Presumably a valued field is real-valued; condition (ii) is stated by the author as $|\langle x, Sy\rangle|=|\langle y, Sx\rangle|$ but he concludes that $(x, y)=(y, x)$; on p. 85, l. 16, d is a coboundary so l. 21 should read "$d_{r+1}d_r=0$".} The group $A_r(\partial)$ of harmonic chains is the intersection of the kernels of ∂ and d and, if $B_r(\partial)$ is the group of ∂-boundaries of C_r and $\Gamma_r(\partial)$ is the group of d-coboundaries, then B_r is completely orthogonal to the group of d-cocycles, Γ_r is completely orthogonal to the group of ∂-cycles, whence, (by suppressed reasoning) $C_r=A_r\oplus B_r\oplus\Gamma_r$. It follows that $A_r(\partial)\simeq H_r(C_\partial)$.

Now suppose that @ is another boundary operator on C and that the image of S is also stable for @; we may thus form a coboundary operator a on C. Suppose further that there are non-zero elements μ_r, ν_r, σ_r, ρ_r of K such that $\mu_r\partial@+\nu_r@\partial=0$, $\sigma_r d@+\rho_r@d=0$, on C_r. It then follows that $A_r(\partial)=A_r(@)$ so that $H_r(C_\partial)\simeq H_r(C_@)$. A host of other representations of $H_r(C_\partial)$ is then deduced, involving combinations of the derivatives ∂, @, d, and a, and the author concludes that, if C' is a subspace of C stable for ∂ and @, and if $C-C'$ is the difference space, then (Theorem 2) $A_r(C)\simeq A_r(C')\oplus A_r(C-C')$. The theory is then applied to exterior differential forms on a compact Kähler manifold ($\mu_r=\nu_r=\sigma_r=\rho_r=1$) and the author obtains from Theorem 2 a count of the number of independent cohomology classes of $H^r(V, M)$ not containing harmonic forms, where V is a compact Kähler manifold and M is an analytic submanifold. {Reviewer's note: the definition of relative harmonic forms is not made explicit.} In the final section it is supposed that the spaces C_r are finite-dimensional and that @ is obtained from ∂ by means of an ("inadmissible") automorphism T of C_r, $@=T^{-1}\partial T$. Necessary conditions for the existence of stable subspaces are inferred. *P. J. Hilton* (Manchester).

Kudo, Tatsuji; and Araki, Shôrô. Topology of H_n-spaces and H-squaring operations. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 85–120.

The authors define a class of Hopf spaces, called H_n-spaces, and show that on the class of H_n-spaces homology operations $Q_i: H_r(X; Z_2)\to H_{2r+i}(X; Z_2)$ $(0\leq i\leq n-1)$ can be defined by a procedure dual to Steenrod's definition of the cohomology operations Sq_i. The prototype of an H_n-space is the n-fold loopspace of a space; in particular an H_0-space is just a Hopf-space. The authors derive the following properties of the Q_i $(1\leq i\leq n-1)$ in $H_*(X; Z_2)$ where X is an H_n-space: (i) Q_i is a homomorphism; (ii) Q_i, $i>0$, is trivial in $H_0(X; Z_2)$; (iii) $Q_0(a)=a^2$ (the square in the Pontryagin ring); (iv) $\Delta_2 Q_i(a_q)=(i+q+1)Q_{i-1}(a_q)$ for any $a_q\in H_q(X; Z_2)$, where Δ_2 denotes the Bockstein operation in homology; (v) the operations Q_i $(0\leq i\leq n-1)$ commute with transgression in the fiber space $\Omega X\to X_1\to X$, whenever X is a simply connected H_n-space. (Here X_1 denotes the space of paths starting at a fixed point.) These are, of course, all dual to corresponding properties of the Sq_i. The authors do not give an analogue of the Cartan formula for $\mathrm{Sq}_i a\cup b$. The paper closes with a very worthwhile application. The operations Q_i enable the authors to dualise Serre's argument for the determination of $H^*(Z_2; n; Z_2)$ (which in turn is based on Borel's transgression theorem). In this manner they determine the mod 2 Pontryagin ring of the Nth loopspace on the n-sphere, whenever $N<n$. *R. Bott* (Ann Arbor, Mich.).

Kelly, Paul J. A congruence theorem for trees. Pacific J. Math. **7** (1957), 961–968.

Let A and B be trees. Let there exist a 1-1 mapping f of the vertices of A onto those of B with the following property: if a is a vertex of A then the graphs obtained from A and B by suppressing a, $f(a)$ and the edges incident with these vertices are isomorphic. Then the author shows that A and B are themselves isomorphic. *W. T. Tutte* (Toronto).

Sedláček, Jiří. Über endliche gerichtete Graphen. Časopis Pěst. Mat. **82** (1957), 195–215. (Czech. Russian and German summaries)

Die vorliegende Arbeit hat vier Teile. Der erste Teil enthält Definitionen der Grundbegriffe: Als unseparabel bezeichnen wir [nach Whitney, Trans. Amer. Math. Soc. **34** (1932), 339–362] einen Graphen, dessen jede zwei Kanten mit einem gemeinsamen Knotenpunkt in irgendeinem Kreis liegen. Die Begriffe Bahn, Kantenbasis, Punktbasis, Netz und ebener Graph sind in König, "Theorie der endlichen und unendlichen Graphen" [Akademische Verlagsgesellschaft, Leipzig, 1936], definiert. Mit $\nu(G)$ bezeichnen wir die Anzahl der Knotenpunkte 2. Grades im Graphen G. Im zweiten Teil studieren wir Kantenbasen für Netze und besonders solche Kantenbasen B, für welche $\nu(B)$ eine kleine Zahl ist. Haupteigenschaften dieser Graphen B sind: Für jedes B ist $\nu(B)\geq 2$ (Satz 4). Ist B ein unseparabler Graph mit gerader (resp. ungerader) Anzahl von Knotenpunkten und $\nu(B)\leq 3$ (resp. $\nu(B)\leq 4$), dann ist B ein ebener Graph (Satz 8). Fig. 5 zeigt einen nichtebenen Graphen B^* mit 10 Knotenpunkten, wo $\nu(B^*)=4$. [Dieser Teil unserer Arbeit knüpft an Rédei, Acta Math. Acad. Sci. Hungar. **5** (1954), 17–25; MR **16**, 58.] Der dritte Teil ist den wohlgerichteten Graphen gewidmet [so bezeichnen wir solche Graphen, in denen von jedem Knotenpunkt aus zu jedem anderen eine Bahn führt — siehe Robbins, Amer. Math. Monthly **46** (1939), 281–283]. Der Begriff

des wohlgerichteten Graphen hängt mit dem algebraischen Begriffe der unzerlegbaren Matrix eng zusammen. Wenn wir nämlich einer gegebenen Matrix $A=\|a_{ik}\|_1^n$ einen Graphen G_A (in weiterem Sinn) zuordnen, der n Knotenpunkte $u_1, u_2, \cdots, u_n$ hat, und dessen Kanten durch die Formel $\overline{u_iu_k} \in G_A \Leftrightarrow a_{ik} \neq 0$ gegeben sind, dann ist A eine unzerlegbare Matrix dann und nur dann, wenn G_A ein wohlgerichteter Graph ist (Satz 9). Ein wohlgerichteter Graph, der h Kanten und u Knotenpunkte hat, enthält wenigstens $h-u+1$ Zyklen (Satz 11).

Der vierte Teil behandelt azyklische Graphen. Ein Graph ist azyklisch, wenn kein Teilgraph ein Zyklus ist. Wenn wir der gegebenen Matrix A wieder einen Graphen G_A zuordnen, so ist G_A azyklisch dann und nur dann, wenn so eine Permutation der Spalten und Reihen der Matrix A existiert, welche die Matrix A in eine Dreiecksmatrix umformt (Anmerkung 5). Weiter wird gezeigt, dass ein azyklischer Graph nur eine einzige Kantenbasis (Satz 13) und eine einzige Punktbasis (Satz 14) enthält. Wenn wir einen vollständigen Graphen als gerichtet definieren, in dem zu jedem Paar verschiedener Knotenpunkte u, v eine und nur eine der Kanten $\overline{uv}$, $\overline{vu}$ existiert, dann gilt der Satz: Ein vollständiger Graph enthält dann und nur dann eine einzige Bahn, die durch alle Knotenpunkte führt, wenn G azyklisch ist (Satz 15).

In der Knotenpunktmenge $\Pi\{G\}$ eines beliebigen gerichteten Graphen G definieren wir die folgende Äquivalenzrelation: Es sei $x\rho y$ dann und nur dann, wenn eine Bahn aus x nach y und zugleich auch eine Bahn aus y nach x führt. Die Relation ρ definiert eine Zerlegung Σ der Menge $\Pi\{G\}$. Die Elemente von Σ können wir für Knotenpunkte eines Graphen $\Sigma\{G\}$ halten, wobei Kanten folgendermassen definiert sind: für $V \neq W$, $V \in \Sigma$, $W \in \Sigma$ existiert $\overline{VW}$ dann und nur dann, wenn es Knotenpunkte $x \in V$, $y \in W$ mit $\overline{xy} \in G$ gibt. Es gilt dann, dass $\Sigma(G)$ ein azyklischer Graph ist. Es wird auch gezeigt, dass man mit Hilfe des Graphen $\Sigma(G)$ jede Punktbasis des Graphen G leicht bilden kann. *Zusammenfassung des Autors.*

Sedláček, Jiří. Über einen extremen ebenen Graphen. Časopis Pěst. Mat. **81** (1956), 426–430. (Czech. Russian and German summaries)

Es seien $a_1, a_2, \cdots, a_k$ gegebene natürliche Zahlen ($k \geq 2$), wobei $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k$, $\sum_{i=1}^{k} a_i = n$. Als eine (zu den Zahlen a_i gehörige) Karte bezeichnen wir jeden zusammenhängenden ebenen Graphen mit n Knotenpunkten ohne Endkanten und ohne Brücken, dessen Knotenpunkte in k Klassen $A_1, A_2, \cdots, A_k$ mit $a_1, a_2, \cdots, a_k$ Elementen eingeteilt sind, wobei keine Kante $\overline{xy}$ für $x \in A_i$, $y \in A_i$ ($1 \leq i \leq k$) existiert. Als Minimalkarte $M_{\min}$ (resp. Maximalkarte $M_{\max}$) bezeichnen wir so eine Karte, die eine minimale (resp. maximale) Anzahl von Kanten besitzt. Es sei $\chi(M)$ die Anzahl von Kanten für die Karte M.

A. Errera [C. R. Acad. Sci. Paris **177** (1923), 489–491] hat bewiesen, dass $\chi(M_{\max}) = 2a_1 + 2a_2 - 4$ für $k=2$ ist. Die vorliegende Arbeit enthält folgende Sätze: 1. $\chi(M_{\min}) = \max(2a_k; n)$. 2. Die der Karte $M_{\max}$ entsprechenden Elementarflächen in der Ebene sind Dreiecke oder Vierecke. 3. Für $k=3$ gilt

$$\chi(M_{\max}) = \min(3a_1 + 3a_2 + 3a_3 - 6;\ 4a_1 + 4a_2 + 2a_3 - 8).$$

Zusammenfassung des Autors.

Collatz, Lothar; und Sinogowitz, Ulrich. Spektren endlicher Grafen. Abh. Math. Sem. Univ. Hamburg **21** (1957), 63–77.

Let the vertices of a connected graph G be $P_1, P_2, \cdots, P_n$. Let a_{ij} be the number of edges joining P_i and P_j. The authors study the eigenvalues of the matrix $A=\{a_{ij}\}$, with special attention to the "simple" case $0 \leq a_{ij} \leq 1$, $a_{ii}=0$. They point out that A has a greatest positive eigenvalue r, and establish the following results. If $q_{\max}$ and $q_{\min}$ are the greatest and least degrees for the vertices of G then $q_{\min} \leq r \leq q_{\max}$. If $\bar{q}$ is the average degree then $\bar{q} \leq r$, with equality if and only if G is regular. If an edge is removed from G without destroying the connection then r is reduced. The characteristic equation of A is tabulated for all simple connected graphs of ≤ 5 vertices, and for all trees of 6, 7 and 8 vertices. In a third table it is given for 42 other simple connected graphs. The tables include the Petersen graph and the graphs of the regular solids.

W. T. Tutte (Toronto, Ont.).

Kotzig, Anton. On certain decompositions of a graph. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 144–151. (Slovak)

In a finite graph G the degree of connectivity $\sigma_G(u, v)$ between vertices u, v is defined as the integer k for which there is a set of k edges, but none of $k-1$ edges, whose removal disconnects u and v. Also, set $\sigma_G(u, u) = +\infty$. The relations $(\overset{k}{=})$, $k=1, 2, \cdots$, defined by $u(\overset{k}{=})v \Leftrightarrow \sigma_G(u, v) \geq k$, are equivalence relations, growing finer with increasing k. Further, we call two edges h_1, h_2 equivalent ($h_1 \sim h_2$) if either $h_1 = h_2$ or every circuit containing h_1 contains h_2. Denote by $T_i (i=1, \cdots, r)$ the equivalence classes of edges. Then $\sigma_G(u, v)=1$ if and only if h is not contained in any circuit (the set of such edges is denoted by $H(1, 2)$; $\sigma_G(u, v)=2$ if and only if h belongs to a certain class T_i containing at least two edges (the set of such edges is denoted by $H(2, 3)$). Further, certain properties of a regular graph of degree n (i.e., one for which every vertex has degree n) are investigated.

The theorem is proved: If a graph G has a decomposition into n linear factors [König, Theorie der endlichen und unendlichen Graphen, Leipzig, 1936, p. 10], then $H(1, 2) = \emptyset$, and for each $i=1, \cdots, r$ all edges of class T_i are contained in the same linear factor. If, further, n is even, then $\sigma_G(u, v)$ is even for any two vertices u, v; if n is odd, then $\sigma_G(u, v)$ is either even or $= n$.

M. Fiedler (RŽMat **1957**, 255).

Kotzig, Anton. Bemerkungen zum Listingschen Satz über die Zerlegung eines Graphen in offene Züge. Časopis Pěst. Mat. **81** (1956), 396–404. (Slovak. Russian and German summaries)

In der Arbeit sind folgende Sätze, die sich dem bekannten Satz von Listing anknüpfen, bewiesen:

1. Es sei G ein endlicher regulärer Graph $(2x+1)$-ten Grades ($x>0$), in dem die Anzahl von Knotenpunkten $2n$ ist und in welchem mindestens ein Faktor ersten Grades L existiert. Dann existiert ein System von offenen Zügen $\mathfrak{Z}=\{Z_1, Z_2, \cdots, Z_n\}$ mit diesen Eigenschaften: a) Jede Kante des Graphen G ist eine Kante genau eines Zuges aus $\mathfrak{Z}$. b) Jeder Zug besitzt dieselbe Anzahl $2x+1$ von Kanten. c) Jeder Zug besitzt eine und nur eine Kante des linearen Faktors L.

2. Es sei G ein endlicher regulärer Graph dritten Grades mit $2n$ Knotenpunkten. Es existiere ein System von offenen Zügen $\mathfrak{Z}=\{Z_1, Z_2, \cdots, Z_n\}$ mit folgenden Eigenschaften: a) Jede Kante des Graphen G ist eine Kante genau eines Zuges aus $\mathfrak{Z}$. b) Jeder Zug besitzt genau drei Kanten. Dann existiert im Graphen G mindestens ein

Faktor ersten Grades. Der analogische Satz für einen Graphen $(2x+1)$-ten Grades mit $x>1$ ist nicht gültig.

3. Man kann einen endlichen regulären Graphen $(2x+1)$-ten Grades G in ein System $\mathfrak{Z}$ von offenen Zügen mit den Eigenschaften a), b) nur dann zerlegen, wenn G keinen Knotenpunkt besitzt, der mit mehr als x Brücken inzidiert. *Zusammenfassung des Autors.*

See also: Combinatorial Analysis: Harary. **Geometry of Numbers:** Reeve. **Harmonic Functions, Convex Functions:** Friedrichs; Morrey and Eells; Morrey. **General Topology:** Munkres. **Differential Geometry:** Segre; Thom. **Manifolds, Connections:** Frankel; Willmore. **Algebraic Geometry:** Morikawa. **Economics, Management Science:** Supnick.

GEOMETRY

Geometries, Euclidean and Other

Neymeyer, L. Das Wesen der Geometrie. Studium Gen. **10** (1957), 292–295.

Guggenheimer, H. Méthodes topologiques en fondements de la géométrie. Bull. Res. Council Israel. Sect. A. **6** (1957), 103–114.

A set π (the plane) of elements $A, B, C, \cdots$ (points) is given together with an order relation subject to the usual axioms of geometry. Triangles are defined and their interiors are taken as neighborhoods. The resulting space is shown to be a completely regular hausdorff space and so admits a uniform structure, hence a completion $\hat{\pi}$. For every pair (A, B) of points in $\hat{\pi}$, a pair (C, D) is said to be as neighborly as (A, B) if it is contained in every vicinity containing (A, B). The first axiom of congruence is imposed in the following form: for every (A, B) and C there is at least one $D\neq C$ such that (C, D) is as neighborly as (A, B). Using a metrization theorem of Cohen and Goffman [Proc. Amer. Math. Soc. **1** (1950), 750–753; MR **12**, 434] in modified form, it is shown that π is metrizable by a totally ordered group G. It is then shown that π is homeomorphic to $G\times G$. *C. Goffman.*

Szász, P. Ein elementargeometrischer Beweis von H. A. Schwarz vereinfacht und unabhängig vom Parallelenaxiom geführt. Acta Math. Acad. Sci. Hungar. **8** (1957), 217–222.

The simplification announced in the title of the paper is based on the following lemma: if two segments $AP=BQ$ lie on opposite sides of a line AB, and the two angles BAP, ABQ are supplementary, then AB is smaller than PQ. The author also points out an extension of Fagnano's proposition in the Euclidean plane to certain polygons with an odd number of sides and makes several pertinent remarks concerning the validity of Fagnano's theorem in the hyperbolic plane. The assertion that there are only two solutions by elementary geometry of Fagnano's problem in the Euclidean plane is an understatement [cf. N. A. Court, Scripta Math. **17** (1951), 147–150; **18** (1952), 95–96]. *N. A. Court* (Norman, Okla.).

Dubikajtis, L. Sur les partages du triangle. Colloq. Math. **4** (1957), 219–223.

The author is concerned with simplicial divisions of triangles into triangles, where any two triangles have at most one point or a whole side in common. Two divisions of a triangle $T=T_1+\cdots+T_n$ and $T=S_1+\cdots+S_n$ are equivalent, if there exists a homeomorphic mapping φ of T onto T such that $\varphi(T_i)=S_i$ $(i=1, \cdots, n)$. The division $T=S_1+\cdots+S_n$ is called a condensation of the division $T=T_1+\cdots+T_k$, if there exist numbers $n_0=0<n_1<\cdots<n_k=n$ such that $T_i=S_{n_{i-1}+1}+\cdots+S_{n_i}$ are, for $i=1, \cdots, k$, divisions of T_i. A division is called simple if it is no condensation of another division. The main result of this paper is that there are four such divisions, to one of which every simple division is equivalent.

J. Łos (Toruń).

Zaddach, Arno. Bemerkungen über spezielle Anti-Fano-Ebenen. Arch. Math. **7** (1957), 425–429.

An anti-Fano plane (AF-plane) is a projective plane in which every quadrilateral has collinear diagonal points (such a plane has also been called a Fano plane by many authors). Gleason has shown [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593] that every finite AF-plane is Desarguesian and hence is coordinatized by a field; furthermore, an AF-plane coordinatized by a near-field R is shown in Pickert [Projektive Ebenen, Springer, Berlin, 1955; MR **17**, 399] to be Desarguesian also, whence R possesses both distributive laws. In the present paper this last result is extended as follows: If R is a Veblen-Wedderburn system with the distributive law $a(b+c)=ab+ac$ (i.e., a quasi-field), coordinatizing an AF-plane, and if the Moufang identity $a((bc)b)=((ab)c)b$ is satisfied for all $a, b, c\in R$, then R satisfies both distributive laws.

D. R. Hughes (Columbus, Ohio).

Hughes, D. R. A class of non-Desarguesian projective planes. Canad. J. Math. **9** (1957), 378–388.

The author proves that the 2nd non-Desarguesian plane of order 9 given by Veblen and Wedderburn [Trans. Amer. Math. Soc. **8** (1907), 379–388] is self-dual and may be characterized by a collineation group of order 78. Moreover, the Veblen and Wedderburn techniques are generalized and this allows the author to construct a new class of non-Desarguesian planes of order p^{2n} for every positive integer n and every odd prime p. Section 2 is concerned with the construction of the class of non-Desarguesian projective planes of order p^{2n}. Let R be a left near-field (not a field) of order p^{2n} containing a field F of order p^n as its center. (The existence of such is assured by the results of Zassenhaus.) Let V denote the set of all ordered triples with components in R, and let A be a certain non-singular linear transformation of V, as a vector space over R. Then the projective plane π is defined in terms of R, F, and A, and the construction is such that π possesses a cyclic collineation group generated by A of order $q=p^{2n}+p^n+1$. The collineation A fixes no point or line of π. Section 3 studies the collineation groups of the planes π of Section 2 in more detail. There is considerable general discussion here as well as a detailed analysis of the plane of order 9. Section 4 introduces coordinates via the planar ternary ring, and this device is used to show that the planes constructed in Section 2 are all non-Desarguesian, whence it follows that the planes cannot be coordinatized by a V-W system in any manner. Apart from the class of projective planes constructed here, all finite planes known are coordinatizable by ternary rings which are at least V-W systems, so that, in this sense, these planes are the weakest known. However, the planes

are of prime power order and they can be coordinatized by a planar ternary ring whose addition forms an elementary abelian group. The paper concludes with unsettled questions for this general class of planes of order p^{2n} which are answered for the plane of order 9.

H. J. Ryser (Columbus, Ohio).

Ostrom, T. G. Transitivities in projective planes. Canad. J. Math. **9** (1957), 389–399.

A collineation which leaves fixed every line through some point p (the center) and every point on some line L (the axis) is a perspectivity. A perspectivity with center on the axis is an elation and one with center not on the axis is a homology. Section 1 is entitled "Uniqueness theorems on perspectivities" and studies elations and homologies. If the group Σ of perspectivities with center p and axis L is transitive on the non-fixed points of each line through p, then the plane is $p-L$ transitive. Section 2 studies the coordinatization of planes admitting two $p-L$ transitive groups. Planes are obtained that may be coordinatized by various types of algebraic systems, depending on the kinds of perspectivities and the relative locations of axes and centers. The author has remarked to the reviewer that Theorem 16 of this section requires the additional restriction in (ii) that $n \neq 9$. Section 3 studies the exceptional cases of the author's previous result on double transitivity. The following theorems are established. Theorem 17: Let π be a finite projective plane of order n, where n is an odd power of two, and let π be doubly transitive. Then π is Desarguesian. Theorem 18: Let π be doubly transitive for points not on the line L_∞, and let n be an odd power of two. Then π is a Veblen-Wedderburn plane. Theorem 19: Let π be finite and transitive on proper quadrangles. Then every quadrangle generates a Desarguesian subplane of prime order μ, where μ divides n or $n-1$. (The author has added in proof that Theorems 17 and 19 are outmoded by the very recent investigations of A. Wagner.) The concluding Section 4 shows that every finite plane contains Desarguesian configurations. *H. J. Ryser* (Columbus, Ohio).

Ostrom, T. G. Double transitivity in finite projective planes. Canad. J. Math. **8** (1956), 563–567.

Let π be a finite projective plane with $n+1$ points per line, where n is odd and not a square. Let S be a set of points of π and assume that the collineation group of π is transitive on the ordered pairs of distinct points belonging to S. The author proves the following theorems: (1) If S consists of all points of π, then π is Desarguesian. (2) If S consists of all points of π not lying on a distinguished line, then π is a Veblen-Wedderburn plane. The essential step in the proof of (1) is to show that, for every point P and line L of π such that P lies on L, the plane π is $P-L$ transitive in the sense of Baer [Amer. J. Math. **64** (1942), 137–152; MR **3**, 179]. This is accomplished by a delightfully elementary study of involutions and perspectivities. The same tools suffice for the proof of (2). *R. H. Bruck.*

Lorent, H. Courbes ou surfaces associées à des faisceaux de courbes ou de surfaces données. An. Fac. Ci. Porto **38** (1954), 81–105.

The author makes an elementary study of some curves and surfaces defined as geometric loci. *P. Abellanas.*

Godeaux, Lucien. Sur un problème de géométrie énumérative. Bull. Soc. Roy. Sci. Liège **26** (1957), 99–104.

This is the extension of a previous paper by the author [Nouvelles Ann. Math. (4) **7** (1907), 395–399]. A linear point connection ("liaison ponctuelle") between the k-tuples of points in projective r-space, S_r, is defined as a relation that is linear separately with respect to the coordinates of each of these points. The author considers n-dimensional linear subspaces of S_r which pass through a linear $(n-1)$-space and intersect $(n+1)^k$ ordered k-tuples of $(r-n)$-dimensional linear subspaces in $(n+1)^k$ ordered k-tuples of points. If the points of each k-tuple are homologous under the same linear point connection, the n-spaces form an $(r-n-1)$-dimensional cone of order $\sum_{i=1}^{k} i \binom{k}{i} n^i$. Application of this theorem to the case $r=3$, $n=k=2$, and further developments yield several numerical results, e.g.: the planes which intersect 10 ordered pairs of straight lines in 10 ordered pairs of points belonging to the same reciprocity represent the envelope of a developable surface of class 76. Similar results are obtained for polarities replacing the reciprocities.

R. Artzy (Madison, Wis.).

★ **Coxeter, H. S. M. Non-Euclidean geometry.** Mathematical Expositions, no. 2., 3rd ed. University of Toronto Press, Toronto, Ont., 1957. xv+309 pp. $5.50.

The 3rd edition of this excellent book differs from the first [published in 1942; MR **4**, 50] only by the following additions, put together as a new chapter. a) A description of the two families of mid-lines or Hjelmslev lines of two given lines, coplanar or skew; b) an elementary derivation of the basic formulae of spherical trigonometry (Napier chain) and hyperbolic trigonometry (Engel chain); c) the computation of the Gaussian curvature of the elliptic and hyperbolic planes when they are referred to isometric or geodesic coordinates; d) a theorem on quadratic forms related with the formulae for distance and angle in non-euclidean geometry (Chap. XII); e) a proof of Schläfli's formula for the differential of the volume of a tetrahedron; f) a brief historical survey of construction problems in non-euclidean geometry by means of ruler, compasses, hypercompass and horocompass, including recent bibliography. *L. A. Santaló* (Buenos Aires).

Szász, Paul. Begründung der analytischen Geometrie der hyperbolischen Ebene mit den klassischen Hilfsmitteln, unabhängig von der Trigonometrie dieser Ebene. Acta Math. Acad. Sci. Hungar. **8** (1957), 139–157.

In terms of orthogonal axes $O\Omega$, OE (with Ω and E at infinity) the author defines the "horocyclic" coordinates (t, σ) of a point P as follows. Let OS and $P'P$ be corresponding horocyclic arcs with the same center Ω. Then t is the length of the straight segment OP' and σ is the length of the horocyclic arc OS. From these he derives the normalized canonical coordinates

$$x_0 = \cosh t + \tfrac{1}{2}\sigma^2 e^{-t},\ x_1 = \sinh t + \tfrac{1}{2}\sigma^2 e^{-t},\ x_2 = \sigma e^{-t},$$

which satisfy $x_0^2 - x_1^2 - x_2^2 = 1$ [cf. the book reviewed above]. He then obtains the corresponding line coordinates X_0, X_1, X_2 (which he calls w, u, v) and shows that two lines $[X]$ and $[Y]$ are intersecting, parallel, or ultraparallel according as the expression $|-X_0Y_0+X_1Y_1+X_2Y_2|$ is less than 1, equal to 1, or greater than 1.

H. S. M. Coxeter (Toronto, Ont.).

Szász, Paul. Die hyperbolische Trigonometrie als Folge der analytischen Geometrie der hyperbolischen Ebene. Acta Math. Acad. Sci. Hungar **8** (1957), 159–161.

The author uses the coordinates defined in the paper

reviewed above to establish the classical relations between the sides and angles of a right triangle [cf. p. 282 of the book reviewed second above]. *H. S. M. Coxeter.*

Fladt, Kuno. Bemerkungen zur Darstellung der ebenen hyperbolischen Geometrie im ebenen euklidischen hyperbolischen Kreisbündel. Acta Math. Acad. Sci. Hungar. 8 (1957), 99–105.

The author derives Lobatschewsky's formula for the angle of parallelism [cf. pp. 207–208 of the book reviewed third above] from the conformal model in which the lines of the hyperbolic plane are represented in the Euclidean plane by circles orthogonal to a fixed circle. His consequent deduction of the classical relations between the sides and angles of a right triangle is notably simpler than that of Eves and Hoggatt [Amer. Math. Monthly **58** (1951), 469–474; MR **13**, 269]. *H. S. M. Coxeter.*

Nádeník, Zbyněk. Quelques propriétés des hyperplans de sommets d'un polygone normal. Časopis Pěst. Mat. **81** (1956), 287–291. (Czech. Russian and French summaries)

A normal polygon in n-dimensional space is formed by $n+1$ linearly independent vertices $A_1, A_2, \cdots, A_{n+1}$ and $n+1$ edges $e_1=A_1A_2, e_2=A_2A_3, \cdots, e_{n+1}=A_{n+1}A_1$. Let β be a hyperplane cutting the line e_i at B_i, let β_i be the hyperplane through B_i and the $n-1$ vertices not lying on e_i, and $'\beta_i$ the hyperplane through these vertices parallel to e_i. The author proves: (i) for odd n, $\beta, \beta_1, \beta_2, \cdots, \beta_{n+1}$ have a common point or a common direction (point at infinity); (ii) for even n, $'\beta_1, '\beta_2, \cdots, '\beta_{n+1}$ have neither a common point nor a common direction; and (iii) for odd n, they have exactly one common direction.

F. A. Behrend (Melbourne).

Nádeník, Zbyněk. Sur le point qui forme une analogie du point de concours des hauteurs d'un triangle pour un polygone normal. Časopis Pěst. Mat. **81** (1956), 292–298. (Czech. Russian and French summaries)

With the notation of the preceding review, let S_i be the midpoint of e_i, and τ_i the hyperplane through S_i and the vertices not lying on e_i. The τ_i always have a point P in common, and the S_i lie in a hyperplane if and only if n is odd (in which case this hyperplane passes through P). If there exists a hypersphere through the S_i, it is called the Feuerbach hypersphere (it always exists for even n). Let V_i be the second point of intersection of the Feuerbach hypersphere with the line e_i (V_i may coincide with S_i), and let ν_i be the hyperplane through V_i and the vertices not lying on e_i. The ν_i may be considered as the "altitudes" of the normal polygon; they possess a common point or a common direction. *F. A. Behrend* (Melbourne).

Lippmann, Horst. Zur Winkeltheorie in zweidimensionalen Minkowski- und Finsler-Räumen. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 162–170.

Let x_1, x_2 be affine coordinates and $F(x_1-y_1, x_2-y_2)$ a Minkowski metric in the affine plane. Besides $F(x_1, x_2)=F(x)>0$ for $x\neq 0$ and $F(\lambda x)=|\lambda|F(x)$, it is assumed that $F(x)$ is for $x\neq 0$ of class C^2 and that $F(x)=1$ has positive curvature. Denote by (x, y) the (not necessarily symmetric) "inner product" $\sum x_iF_{x_i}(y)/F(x)$ of x and y in Finsler's sense. The principal result is this: If a is transversal to b, then $f(s)=(b, a+sb)$ has the following properties: $f(s)$ is of class C^1, $f(0)=0$, $f'(s)>0$, $\lim_{s\to\pm\infty} f(s)=\pm 1$, $\lim_{|s|\to\infty} |s|^3f'(s)$ exists and is different from 0. Conversely, if two independent vectors a, b, a $B>0$, and a function $f(s)$ with the above properties are given, then exactly one $F(x)$ with $F(b)=B$ exists for which a is transversal to b and $(b, a+sb)=f(s)$. For $x=\lambda(a+sb)\neq\rho b$ the function $F(x)$ is given by $F(x)=|\lambda|\{F(b)\int_0 f(s)ds+F(a)\}$.

H. Busemann.

See also: Fields, Rings: Stöcker. **Approximations, Orthogonal Functions:** Paszkowski. **Lie Groups and Algebras:** Tits. **Convex Domains, Integral Geometry:** Linis. **Algebraic Geometry:** Barlotti.

Convex Domains, Integral Geometry

Linis, Viktors. Ovals with equichordal points. Amer. Math. Monthly **64** (1957), 420–422.

This is an attempt to demonstrate that no closed convex curve can have two distinct equichordal points, i.e., points such that the lengths of all chords through each one of them are equal. (This is an old problem with which the names of Blaschke, Rademacher and Süss are associated.) The present proof is invalidated by a mistake on p. 422, line 19. The statement that $R'(\beta_1(\alpha))=0$ if and only if $R'(\beta_2(0))=0$ does not follow from what precedes and is in fact untrue (as a property of such curves) if $a\neq 0$ and $0°<\alpha<90°$. Also the statement on p. 422, line 6, that $R(\beta)$ is strictly increasing in an interval $[0, \beta_0]$, $\beta_0>0$, is not shown to follow from the definition of the curve and amounts to further requirement. But because of the mistake just pointed out the non-existence of closed convex curves with two equichordal points satisfying this additional requirement is not proved. *G. A. Dirac.*

Kelly, P. J.; and Straus, E. G. Inversive and conformal convexity. Proc. Amer. Math. Soc. **8** (1957), 572–577.

Let S be a set with interior points in euclidean n-space. Consider the $(n-1)$-spheres and -planes K through a boundary point x of S with the property that S lies in the closure of one side of K. They are "extremal" if that side is minimal. The set K_x is the union of the closures of the other sides of all the extremal K's through x. The authors prove: There is an inversion with the centre p mapping S onto a convex set if and only if $p\in\bigcap_x K_x$. They also describe the loci of those points that are centres of inversions which transform a given plane curve into a convex (respectively locally convex) one. The following corollary of their main results appears particularly attractive. Let C be a closed convex twice differentiable plane curve. Then the intersection of the closed interiors of the circles of curvature of C is identical with the intersection of the closed interiors of the maximal inscribed circles of C.

P. Scherk (Saskatoon, Sask.).

★ **Goldman, A. J.; and Tucker, A. W. Polyhedral convex cones.** Linear inequalities and related systems, pp. 19–40. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

Systematic exposition of the fundamental facts on convex polyhedral cones in n-space: their face structure, the effect of polarity, the lattice of all such cones by set-inclusion. *T. S. Motzkin* (Los Angeles, Calif.).

Heller, I. Neighbor relations on the convex of cyclic permutations. Pacific J. Math. **6** (1956), 467–477.

For a convex polyhedron P with vertices p_i, let k_{ij} be the dimension of the smallest face containing p_i and p_j,

and $K=\max k_{ij}$. If the p_i are the $n!$ points in n^2-space given by square matrices representing permutations of n elements, dim $P=(n-1)^2$ and [Heller, Proc. 2nd Symposium Linear Programming, Washington, D.C., 1955, Nat. Bur. Standards, 1955, pp. 643–665; MR 17, 873] $K=[n/2]$. If the p_i are the $(n-1)!$ points corresponding to cyclic permutations, it is shown that $K=[n/2]-1$ for $n=4$, 5, 7, or $4m+2$, and $K=[n/2]$ otherwise.

T. S. Motzkin (Los Angeles, Calif.).

Pleijel, Arne. Zwei kennzeichnende Kreiseigenschaften. Arch. Math. 7 (1957), 420–424.

Let K be a closed plane convex curve and let $s(\alpha)$ denote the arc length corresponding to the point whose supporting line forms the angle α with a fixed direction θ. Let r be the length of the chord connecting the points of contact of the supporting lines at $s(\alpha)$, $s(\beta)$ $(\alpha<\beta)$ and let φ be the angle the chord makes with the direction θ. Then

$$\int_0^{2\pi} ds(\alpha)\int_\alpha^{\alpha+\pi} r^n ds(\varphi) - \frac{n+2}{n+1}\int_0^{2\pi} ds(\alpha)\int_\alpha^{\alpha+\pi} r^n \sin u \sin v\, ds(\varphi) \geqq 0,$$

where $u=\varphi-\alpha$, $v=\beta-\varphi$, $n\geqq 0$. The equality holds only for the circle. Another inequality of the same type is also given.

L. A. Santaló (Buenos Aires).

★ **Masotti Biggiogero, Giuseppina. Sulla geometria integrale: nuove formule relative agli ovaloidi.** Scritti matematici in onore di Filippo Sibirani, pp. 173–179. Cesare Zuffi, Bologna, 1957.

Let K be a closed convex surface with continuous principal radii of curvature at each point. To each exterior line R corresponds the secant line S which unites the contact points A_1, A_2 of the tangent planes to K through R. Let k_1, k_2 be the Gaussian curvatures of K at A_1, A_2; σ the length of the chord A_1A_2; and ω the angle between the two tangent planes. Then the densities for R and S are related by $dR=(k_1k_2\sigma^4/\sin^4\omega)dS$. From this result some integral formulas are deduced; for instance,

$$\int(\sin^4\omega/k_1k_2\sigma^4)dR=(\pi/2)F,$$

$$\int(\sin^4\omega/k_1k_2\sigma^3)dR=2\pi V,$$

$$\int(\sin^4\omega/k_1k_2)dR=6V^2,$$

where F is the area of K, V the enclosed volume, and the integrals are extended over all straight lines R exterior to K.

L. A. Santaló (Buenos Aires).

See also: Topological Vector Spaces: Phelps.

Differential Geometry

Haupt, Otto. Über Kennzeichnungen lokal konvexer, ebener Bogen, insbesondere auch dualisierbarer. Abh. Math. Sem. Univ. Hamburg 21 (1957), 44–54.

In this paper, results by Unger [Elem. Math. 8 (1953), 79–85; Arch. Math. 4 (1953), 143–153; MR 15, 251] are made precise; they are derived from general theorems due to A. Rosenthal [Math. Ann. 73 (1913), 480–521] and supplemented by original results.

Given a continuous mapping $\mathfrak{P}=\mathfrak{P}(t)$ of a closed interval $\mathfrak{T}$ onto the set $\mathfrak{B}$ in euclidean plane. The arc B is the set of all the "Stellen" $P(t)=(t, \mathfrak{P}(t))$. It is called simple if the mapping is topological. The (one-sided) neighbourhoods on $\mathfrak{T}$ define those on B. If t has right and left neighbourhoods with the same images on $\mathfrak{B}$, $P(t)$ is called an N-spot ("Nadelstelle"). The arc B is locally convex at $P(t)$ if some neighbourhood of $P(t)$ is a simple subarc of B and if the subset of $\mathfrak{B}$ determined by this subarc lies on the boundary of its convex hull. If t' and t'' converge to t $(t'\neq t'')$, any limit of straight lines through $\mathfrak{P}(t')$ and $\mathfrak{P}(t'')$ is called a paratingent of B at t.

Let B be an arc without N-spots. If it has no parallel paratingents, then it is simple and $\mathfrak{B}$ lies on the boundary of its convex hull. Also the following properties of B are equivalent: (a) For every t, $\mathfrak{P}(t)$ has a neighbourhood on $\mathfrak{B}$ which lies on the boundary of its convex hull. (b) Let t' and t'' converge to any t; $t'\neq t''$. Then the intersection of any paratingent at t' with any paratingent at t'' converges to $\mathfrak{P}(t)$. (c) Every $P(t)$ has a neighbourhood without parallel paratingents.

From now on assume (i) if t' converges to any t, any straight line through $\mathfrak{P}(t)$ and $\mathfrak{P}(t')$ converges; (ii) its limit $p(t)$ depends continuously on t; and (iii) if t' converges any t, $p(t)\cap p(t')$ converges to $\mathfrak{P}(t)$.

Let t lie in the interior of $\mathfrak{T}$. Choose a point $\mathfrak{A}\neq\mathfrak{P}(t)$ and a straight line $a\neq p(t)$. Let t^+ $[t^-]$ range through a small right [left] neighbourhood of t. The following conditions define semiregularity [regularity] of B at t: I) If the lines $\mathfrak{A}\mathfrak{P}(t^+)$ and $\mathfrak{A}\mathfrak{P}(t^-)$ are different from $\mathfrak{A}\mathfrak{P}(t)$, they are separated by $\mathfrak{A}\mathfrak{P}(t)$ if $\mathfrak{A}\notin p(t)$ [and not separated if $\mathfrak{A}\in p(t)$]. II) If the points $a\cap p(t^+)$ and $a\cap p(t^-)$ are different from $a\cap p(t)$, they are separated by $a\cap p(t)$ if $\mathfrak{P}(t)\notin a$ [and not separated if $\mathfrak{P}(t)\in a$]. The following properties of B are equivalent: (i) B is locally convex every where. (ii) B is regular everywhere. (iii) B is semiregular everywhere. If in addition B has no N-spots, these properties are equivalent to the property (iv): B has a unique paratingent everywhere and satisfies (b).

P. Scherk (Saskatoon, Sask.).

Bouligand, Georges. Une classe de surfaces localement constructibles. C. R. Acad. Sci. Paris 244 (1957), 2212–2214.

The functions $f(x, y)$ and $g(x, y)$ are defined and convex in a convex domain of the (x, y)-plane. The author makes comments on the direct differential geometry of the surface $Z=f(x, y)-g(x, y)$ and proposes various problems.

P. Scherk (Saskatoon, Sask.).

★ **Mihăileanu, N. Sur l'espace V_m plongé dans un E_n.** Travaux de la conférence de géometrie différentielle du 9 au 12 juin 1955, pp. 161–167. Editura Academiei Republicii Populare Romîne, Timişoara. (Romanian. Russian and French summaries)

Let $M(u^1, u^2, \cdots, u^n)$ be an arbitrary point in a euclidean space E_n referred to curvilinear coordinates u^i and let Γ_{jk}^i be the Christoffel symbols of the second kind in this coordinate system. From the relation $\partial^2M/\partial u^j\partial u^k=(\partial M/\partial u^h)\Gamma_{jk}^h$ the author obtains

$$(*)\qquad \Gamma_{jk}^h=\frac{\left(\frac{\partial M}{\partial u^1}\cdots\frac{\partial M}{\partial u^{h-1}}\,\frac{\partial^2 M}{\partial u^j\partial u^k}\,\frac{\partial M}{\partial u^{h+1}}\cdots\frac{\partial M}{\partial u^n}\right)}{\left(\frac{\partial M}{\partial u^1}\cdots\frac{\partial M}{\partial u^n}\right)}.$$

This formula is then used to derive in a simple way the equations of Gauss, Codazzi and Ricci-Kuehne for a V_m imbedded in an E_n. This is done in the following way, indicated by Vranceanu [Atti Accad. Naz. Lincei. Rend.

Cl. Sci. Fis. Mat. Nat. (6) **11** (1930), 385–389]. Starting from a formula which expresses the metric of the E_n in terms of the metric of the imbedded V_m, the components of the Riemannian curvature tensor of the E_n are determined with the help of (*). The vanishing of these components on the V_m results in the required equations.

R. Blum (Saskatoon, Sask.).

Mihăileanu, N. Submersion d'une surface dans un espace non euclidien. Com. Acad. R. P. Romîne **6** (1956), 757–762. (Romanian. Russian and French summaries)

If $1/h$ is the total curvature of a non-euclidean space of three dimensions (S_3), there exists a 'preferred' system of homogeneous coordinates x_i in which its metric is $ds^2=hdx_0^2+dx_1^2+dx_2^2+dx_3^2$. The author interprets this as the metric of a euclidean space of four dimensions and uses formula (*) of the previous review and the method indicated there in order to obtain the equations of Gauss and Codazzi of a surface imbedded in S_3. *R. Blum.*

Grigor'ev, I. N. Asymptotic transformation of p-orthogonally conjugate systems in n-space. Moskov. Gos. Univ. Uč. Zap. **181**. Mat. **8** (1956), 91–106. (Russian)

Bianchi has shown that a triply orthogonal system, containing one family of surfaces of constant negative curvature allowing a two-parametric family of transformations, satisfies the following properties: 1) a congruence ray touches correspondent pseudo-spherical surfaces, 2) the asymptotic lines on the focal surfaces correspond, 3) the lines of intersection of the surfaces of the given system pass into the lines of intersection of the transformed system of surfaces (they are lines of curvature). On the other hand, Finikov has shown in his "Theory of congruences" [Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 744] that there exists a wider class of W-congruences which transfer the lines of curvature of the given surfaces into the lines of curvature of the transformed surfaces. It is now shown that only pseudo-spherical triply orthogonal systems allow a one-parametric family of transformations which satisfy all three properties 1), 2), 3). Such transformations are here called asymptotic. The first part of the paper is devoted to a proof of this proposition. In the latter part it is demonstrated that a triply orthogonal system of three one-parametric families of two-dimensional surfaces in n-space ($n>3$) exists, depending on $(n-3)$ functions of three variables. A p-orthogonal-conjugate system in n-space, satisfying asymptotic transformation, consists of a $(p-3)$-parametric family of congruent pseudo-spherical triply orthogonal systems, every one of which lies in one 3-space. Every such triply orthogonal element of the system is transformed by means of pseudo-spherical congruences into a triply orthogonal element of the same type lying in that 3-space. [See also Grigor'ev, Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 765–767; MR **16**, 621.] *D. J. Struik.*

Nitsche, Johannes. Über eine mit der Minimalflächengleichung zusammenhängende analytische Funktion und den Bernsteinschen Satz. Arch. Math. **7** (1957), 417–419.

Let $z(x, y)$ be a twice continuously differentiable solution of the partial differential equation of minimal surfaces

$$(1+q^2)r-2pqs+(1+p^2)t=0 \tag{1}$$

in a convex domain G of the xy-plane. The author introduces new variables u, v by means of the formulas

$$u=x+\int_{(x_0,y_0)}^{(x,y)} W^{-1}[(1+p^2)dx+pqdy], \tag{2}$$

$$v=y+\int_{(x_0,y_0)}^{(x,y)} W^{-1}[pqdx+(1+q^2)dy], \tag{3}$$

where (x_0, y_0) is a point in G and $W=(1+p^2+q^2)^{\frac{1}{2}}$ (note that the integrands in (2) and (3) are complete differentials in view of (1)). He verifies the fact that (2) and (3) define a 1-1 mapping from G into the uv-plane such that the distance of no pair of points in G is diminished by the mapping. Hence, if G coincides with the whole xy-plane, then the mapping is onto the whole uv-plane. The author verifies further that on setting $u+iv=w$, the function $F(w)=(p-iq)(1+W)^{-1}$ is an analytic function of w in the image domain. Since obviously $|F(w)|<1$, one obtains a surprisingly simple proof of the famous theorem of S. Bernstein that if G coincides with the whole xy-plane, then $z(x, y)$ is a linear function of x and y. Indeed, in this case $F(w)$ is a bounded analytic function in the whole w-plane, and hence $F(w)$ reduces to a constant, and consequently p and q also reduce to constants. The author indicates that in terms of the function $F(w)$ one can derive in a very simple manner various estimates for the Gauss curvature of the minimal surface $z=z(x, y)$ which were obtained by other workers by somewhat devious methods.

T. Radó (Columbus, Ohio).

Özkan, Asim. Über die Sechseckbedingungen bei einer n-Kurvenwabe in der Ebene. Abh. Math. Sem. Univ. Hamburg **21** (1957), 95–98.

The paper deals with n-webs of curves in the plane with the property that any 3-subweb satisfies the hexagon condition. There are $\binom{n}{3}$ conditions. It was known previously that for $n=4$ the 4 conditions are independent and that 9 of the 10 conditions are independent for $n=5$ [W. Blaschke and G. Bol, Geometrie der Gewebe, Springer, Berlin, 1938, p. 108]. It is shown that in the general case $(n-2)^2$ of the $\binom{n}{3}$ conditions are independent and the remaining ones are redundant. *H. Busemann.*

Jha, P. On binary quadratic forms in geometry. Ganita **7** (1956), 7–11.

Given two binary quadratic differential forms f and ϕ, the author states the condition obeyed by the coefficients of the forms for the orthogonality relative to f of the two families of curves for which $\phi=0$. In particular, the curves for which the Jacobian of ϕ and f is zero form an orthogonal net. Using these and similar results, by specialization of the forms ϕ and f, the author derives several known results concerning rectilinear congruences as well as some new theorems. *A. Fialkow.*

Karapetyan, S. E. Closed cycle of four congruences. Mat. Sb. N.S. **41(83)** (1957), 177–194. (Russian)

In einer früheren Arbeit des Verf. [Sb. Naučn. Trudov. Armyan Ped. Inst. Abovyan. Nr. **5** (1955)] traten 4 Kongruenzen des P_3 auf, die einen Zyklus bilden von der Art, daß die Gerade einer Kongruenz die entsprechende der folgenden in einem Brennpunkt schneidet. In der vorliegenden Arbeit werden nun viele Sonderfälle eines solchen Kongruenzenquadrupels genauer untersucht. Als Hilfsmittel wird der Cartansche Kalkül der den Kongruenzen angepaßten Differentialformen benutzt, wie

er für diesen Zweck von Finikov entwickelt wurde [Die Methode der äußeren Formen von Cartan···, Gostehizdat, Moscow-Leningrad, 1948; Projektiv-differentiale Geometrie, ibid. 1937; MR **11**, 597]. Die behandelten Sonderfälle sind vor allem folgende. a) Alle 4 Kongruenzen sind parabolisch. b) Ein Paar gegenüberliegender Kongruenzen ist schichtbar, d.h., diese bilden ein sog. T-Paar; liegt auch der Fall a) vor, so ist auch das andere Paar schichtbar. c) Der Zykel von 4 zusammengehörigen Strahlen bestimme das räumliche Viereck $A_1 \cdots A_4$, derart daß A_1A_2, A_2A_3, $\cdots$ die 4 Strahlen sind; dann bilden die Geraden A_1A_3, A_2A_4 die sog. Diagonalkongruenzen, und es liegt die Konfiguration Θ_{13} vor, wenn die Punkte A_1, A_3 für die Kongruenz der Geraden A_1A_3 Brennpunkte sind. Es wird die Existenz einer Konfiguration Θ_{13} gezeigt, insbesondere auch wenn ein T-Paar dabei vorkommt. d) Beide gegenüberliegenden Paare von Kongruenzen sind T-Paare; dann entsprechen sich die Torsen derselben wechselseitig. Es wird gezeigt wie und mit welcher Willkür man zu einem solchen Paar gegenüberliegender Kongruenzen ein weiteres Paar konstruieren kann so, daß insgesamt ein geschlossener Viererzyklus entsteht.

W. Burau (Hamburg).

Kamalamma, K. N. Note on a generalized Ribaucour congruence. Math. Student **24** (1956), 230–232 (1957).

La définition des congruences de Ribaucour met en jeu une correspondance par éléments linéaires entre deux surfaces S et $\bar{S}$; les rayons de la congruence sont issus des différents points de S parallèlement aux normales à $\bar{S}$ aux points correspondants. L'auteur envisage les congruences obtenues par le procédé précédent lorsque l'orthogonalité des éléments linéaires de S et $\bar{S}$ n'est plus exigée. Il donne à ces congruences le nom de congruences de Ribaucour généralisées, et étudie quelques-unes de leurs propriétés.

P. Vincensini (Marseille).

Gallissot, François. Les formes extérieures et la mécanique des milieux continus. C. R. Acad. Sci. Paris **244** (1957), 2347–2349.

The author indicates how the equations of continuum mechanics, for some simple cases, can be formulated in the abstract language of modern differential geometry. In each case the basic equations are generated by an exterior differential form over a certain manifold of "jets" [C. Ehresmann, Géométrie différentielle, Colloques Internat. Centre Nat. Rech. Sci. Strasbourg, 1953, pp. 97–110; MR **16**, 75]. *W. Noll* (Pittsburgh, Pa.).

★**Segre, Beniamino. Forme differenziali e loro integrali. Vol. II. Omologia, coomologia, corrispondenze ed integrali sulle varietà.** Edizioni Universitarie Docet, Roma, 1957. 422 pp. (mimeographed) 4500 Lire.

[For the first volume (1951) of this work see MR **14**, 208.] The three chapters of this second volume are entitled: Algebraic preliminaries, Topological preliminaries, Integrals on differentiable manifolds. Most of the material is standard and leads up to the de Rham theorems in Chapter III. There is however one section which is novel. This deals with the theory of correspondences between differentiable manifolds, and is developed at some length. The author's object seems to be to extend, as far as possible, results on correspondences between algebraic varieties. Applications of this theory are given to results such as the Borsuk-Ulam theorem concerning antipodal points on spheres. *M. F. Atiyah.*

★**Thom, R. Approximation algébrique des applications différentiables.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 5 pp.

The author's main result is the following theorem. Every compact, differentiable sub-manifold of a real algebraic variety can be deformed into an algebraic subvariety by an arbitrarily small isotopic deformation (here "real algebraic variety" means "connected component without singular points of a real affine variety"). A corollary of this result is the following theorem, due originally to John Nash [Ann. of Math. (2) **56** (1952), 405–421; MR **14**, 403]: Every compact differentiable manifold can be given the structure of a real algebraic variety (in the above sense). Moreover, the algebraic variety structure which is an extension of the given differentiable structure is essentially unique.

In his proof, the author makes essential use of methods he developed in a previous paper [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890]. *W. S. Massey.*

Decuyper, Marcel. Quadrilatère de Demoulin d'une surface. Rend. Sem. Mat. Messina **1** (1955), 120–142.

The author submits a unified treatment concerning the envelope of the quadrics of Lie of a given surface S in a Euclidean space E_3 of three dimensions. In the general case, at a given point M of S, the quadric of Lie defines what is termed the quadrilateral of Demoulin. A brief biography is given of the Belgian mathematican, Alphonse Demoulin, who was born in 1869 and died in 1947. Many of the theorems of Demoulin and the various co-workers are presented in this article. The methods used are those whereby any line in E_3 is represented by the Plücker coordinates p. Thus any line in E_3 can be depicted as a point P on the four dimensional quadric Q of Klein which is immersed in a projective space S_5 of five dimensions. If (U, V) denote the asymptotic tangent lines of a given surface S of E_3, then S is visualized as a congruence of lines in Q. From this are deduced some theorems concerning the sequence of Lie associated with any such surface S. Next are studied the directrices of Wilczynski and the quadric of Lie. Some properties of the moving tetrahedron of E. Cartan, which can be associated at any point M of the surface S, are given. Of course, many of the properties of the quadrilateral of Demoulin are proved. Finally, many other problems associated with this quadrilateral are discussed. *J. De Cicco* (Chicago Ill.).

★**Villa, Mario. Ancora sui riferimenti intrinseci per le trasformazioni puntuali in una coppia a Jacobiano nullo.** Scritti matematici in onore di Filippo Sibirani, pp. 343–345. Cesare Zuffi, Bologna, 1957.

The reviewer has examined a point-transformation T between two projective planes in the neighborhood of a pair of points where the Jacobian vanishes, determined the rational transformations (1, 2) having a 3rd order contact with T, and using them has given a canonical form of T up to the terms of the 4th order [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14** (1943), 11–21; MR **8**, 219].

With a slight change of the intrinsically determined reference system the author finds a different (but not simpler) canonical form. *E. Bompiani* (Rome).

Bompiani, E. Trasporto di elementi di 2° ordine in un piano proiettivo. Abh. Math. Sem. Univ. Hamburg **21** (1957), 92–94.

In non-homogeneous point coordinates $P(x, y)$ of a

projective plane π, consider a one-parameter family F of third order differential elements E_3: $y=x^2+\lambda x^3+[4]$, all of which contain the fixed second order differential element E_2^*: $y=x^2+[3]$. Let E_1 and E_1' be two lineal elements with the two centers $P(x, y)$ and $P'(x', y')$, and the two tangent lines: $Y-y=t(X-x)$ and $Y-y'=t'(X-x')$, such that $y\neq 0$, $y\neq tx$, and $y'\neq 0$. There is one and only one E_3 of F for which E_3 and E_1 belong to the same conic section. If the E_1' is on this E_3, then E_1' is said to be parallel to E_1. The displacement of E_1 into E_1', is given as a bilinear expression in t and t' for which the coefficients are polynomials in (x, y, x', y'), of maximum degree four. This displacement induces a projectivity between the two pencils of lines with vertices at the two points P and P'. The infinitesimal parallel displacement is linear in the components of the direction of displacement but not in the components of the direction that is being displaced. If π is supposed to be a Euclidean plane, the infinitesimal parallel displacement is written in polar coordinates. In the final part of the paper, the author shows how to use this transformation in order to discuss the parallel displacement of a second order differential element E_2 into another second order differential element E_2'. By means of the coordinates of Study-Engel, such a parallel displacement is found in polar coordinates.

J. De Cicco (Chicago, Ill.).

Picasso, E. Su particolari correlazioni definite dagli iperpiani cuspidali di una superficie non parabolica di S_4. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 147–155.

Given a surface in a projective space S_4 with a conjugate net, at each of its points x are attached two cuspidal hyperplanes S_3 (containing the tangent plane and the osculating plane to one curve of the net at x). These cuspidal S_3's describe two systems whose Laplace's equations are transformed into each other. By means of these systems it is possible to determine two polarities at x or two quadrics defining them. The invariant character of these quadrics is strictly related to the existence of a projective connexion on the surface already introduced by the author [same Rend. **23** (1953), 1–8; MR **15**, 826].

E. Bompiani (Rome).

Lennes, G.; et Rozet, O. Sur les congruences W. Bull. Soc. Roy. Sci. Liège **25** (1956), 341–346.

Let (x) and $(\bar{x})$ denote the two focal surfaces of a W-congruence in a projective space of three dimensions. Let (x) and $(\bar{x})$ be referred to asymptotic parameters (u, v) with the projective homogeneous coordinates of the generating points x and $\bar{x}$ satisfying Wilczynski systems of partial differential equations (I) and $(\bar{I})$, respectively. Let $\{J\}$ denote the set of eight fundamental projective invariants of the surface (x), and $\{\bar{J}\}$ the corresponding set of invariants of $(\bar{x})$. In the present note each invariant of the set $\{\bar{J}\}$ is expressed algebraically in terms of invariants among the set $\{J\}$ and their partial derivatives.

P. O. Bell (Palo Alto, Calif.).

Ščerbakov, R. N. Projective-differential reference frames of a ruled surface belonging to a given congruence. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 390–393. (Russian)

The corresponding affine theory was developed in Mat. Sb. N.S. **37(79)** (1955), 527–556 [MR **17**, 1127], the metrical theory in Uč. Zap. Buryat-Mongol. Gos. Ped. Inst. **1954**, no. 5, 61–89. To establish reference tetrahedra $A_1A_2A_3A_4$ in the present case, A_1A_2 is taken along a ray of the congruence, $dA_i=\omega_i{}^kA_k$, $\omega_2{}^3=\omega_1{}^4$, $\omega_2{}^4=\omega_1{}^3$; the foci F_i are given by $F_1=A_1+A_2$, $F_2=A_1-A_2$, the developables of the congruences by $(\omega_1{}^3)^2-(\omega_1{}^4)^2=0$. The choice is further narrowed by taking planes $A_1A_2A_3$, $A_1A_2A_4$ as tangent planes at A_1, A_2 to two ruled surfaces conjugate in the sense of Sannia. To complete the determination of the frames two possibilities are indicated, both based on transformations of Laplace of the congruence. The final set of equations with integrability conditions consists of 11 external differential forms in 14 functions and determines the congruence with respect to an arbitrary set of conjugate linear surfaces, with 3 arbitrary functions of 2 variables. The computation is indicated, the geometrical meaning of certain invariants is explained and it is shown how natural equations of projectively invariant classes of ruled surfaces can be obtained. The paper ends with a projective interpretation of the transformation of Egorov.

D. J. Struik.

Kovančov, N. I. Quasi-special complexes. Mat. Sb. N.S. **41**(83) (1957), 333–360. (Russian)

A line complex determines in every plane a curve S which is the envelope of the complex lines in that plane. In the quasi-special case there exists at least one surface σ whose tangent planes contain complex lines which degenerate into pencils, so that the curve S becomes a point M. Examples are those complexes consisting of tangents to a surface. In general, however, the points M need not lie on σ itself. This leads to those complexes with multiple inflexion points which are met in Kovančov, Mat. Sb. N.S. **38(80)** (1956), 107–128 [MR **18**, 413]. Indeed, for such complexes the manifold of inflexion centers degenerates into a surface, and every set of rays corresponding to such center degenerates into a plane pencil of lines. In the case of two double inflexion centers on each ray there are four cases called those of complexes K_1, K_2, K_3, K_4. For K_1 there is one and only one stratification into a two parametric multiplicity of plane pencils, for K_2, K_3, K_4 there are respectively 2, 3, 4 such stratifications. All four cases are carefully investigated.

D. J. Struik (Cambridge, Mass.).

Kallenberg, G. W. M. Differential geometry of a particular group of projective transformations. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 147–158.

L'auteur étudie quelques problèmes de géométrie différentielle d'un groupe d'ordre six dans l'espace à trois dimensions. En coordonnées homogènes, l'absolu est constitué par le plan $x_4=0$, la droite $x_3=x_4=0$ et le point $(1, 0, 0, 0)$. En coordonnées non homogènes, la "distance" de deux points $P_1(x_1, x_2, x_3)$, $P_2(y_1, y_2, y_3)$ pris dans cet ordre est y_3-x_3; l'"angle" des deux vecteurs (l_1, l_2, l_3), (m_1, m_2, m_3) dans cet ordre est $\varphi=m_2/m_3-l_2/l_3$; ces définitions sont complétées par celles de distances et d'angles d'ordre supérieur. Une courbe gauche est définie à un "déplacement" près par deux fonctions qu'on appelle courbure et torsion. Une théorie des surfaces est établie au moyen d'une "fonction caractéristique" et l'auteur trouve, dans cette géométrie, des formules de Gauss, un théorème de Meusnier et des indicatrices de Dupin.

M. Decuyper (Lille).

Pan, T. K. Surfaces in a conformal correspondence. Proc. Amer. Math. Soc. **8** (1957), 563–571.

The author studies two surfaces S and $\bar{S}$ in conformal

correspondence. Given two unit vector fields v and $\bar{v}$ on S and $\bar{S}$ respectively, he defines a "curve of v" on S as a curve on S to which the vectors of the field v are tangent. If the "curve of v" happens to be a curve on S along which the vectors v are parallel in Levi-Civita's sense, then the curve is called the indicatrix of v on S. The magnitude of the normal component of the derived vector of v on a curve of S is called the normal curvature of v with respect to the curve.

In a conformal correspondence between S and $\bar{S}$ the author studies the existence of surfaces for which there is correspondence between (a) the curves of v and $\bar{v}$ and (b) the indicatrices, and for which there is equality between the normal curvatures. He uses Elie Cartan's method of moving trihedrals. His main theorem is that there exist surfaces in conformal correspondence preserving the curves, indicatrices and normal curvatures of associated unit vector fields. The general solution depends on six arbitrary functions of one variable. He also gives the singular solution of the problem. He ends with a discussion of the Cauchy problem relative to the general solution.

E. T. Davies (Southampton).

See also: Convex Domains, Integral Geometry: Kelly and Straus. **Algebraic Geometry:** Kirby. **Optics, Electromagnetic Theory, Circuits:** Kohler.

Manifolds, Connections

Palais, Richard S. On the differentiability of isometries. Proc. Amer. Math. Soc. **8** (1957), 805–807.

$\mathfrak{M}$ sei eine n-dimensionale Mannigfaltigkeit der Klasse C^k, wobei $k=1, 2, \cdots\infty, \omega$. Auf $\mathfrak{M}$ liege eine Riemannsche Struktur vor, deren definierendes Tensorfeld bei passender Wahl der Koordinatensysteme zur Klasse C^{k+1} gehört, und M sei der entsprechende metrische Raum. Der Verf. zeigt, daß $\mathfrak{M}$ kanonisch mit Hilfe von M allein rekonstruiert werden kann, und verschärft so ein Ergebnis von Myers und Steenrod [Ann. of Math. (2) **40** (1939), 400–416], wonach jeder Automorphismus von M auch einer von $\mathfrak{M}$ ist. Die Konstruktion beginnt, bei beliebigem x aus M, mit der Beschreibung des Raumes M_x der Geodätischen in x, der mit der Struktur eines n-dimensionalen euklidischen Raumes (dem Tangentialraum von $\mathfrak{M}$ in x analog) versehen wird. Eine Umgebung von 0 in M_x läßt such dann durch eine nichtsinguläre C^k-Abbildung umkehrbar eindeutig auf eine Umgebung von x in $\mathfrak{M}$ abbilden.

K. Krickeberg (Würzburg).

Kaul, R. N. On the magnitude of the derived vector of the unit normal to a hypersurface. Tensor (N.S.) **7** (1957), 70–72.

L'auteur établit une expression simple pour la longueur k_1 du vecteur dérivé du vecteur unitaire normal à une hypersurface V_n, d'un espace Riemannien à $n+1$ dimensions, le long du vecteur tangent à une courbe de V_n. Il en déduit quelques résultats, et donne une généralisation de k_1 pour les sous-espaces des espaces Riemanniens.

P. Vincensini (Marseille).

Kaul, S. K. On a family of curves in a family of subspaces of a Riemannian space and their deformations. Matrix Tensor Quart. **7** (1957), 100–106.

The author considers a Riemannian manifold V_n with a leaved or laminated structure defined by a family of subspaces $\{V_m\}$, $n>m\geqq 1$. He studies curve families $\{C_m\}$ whose individual curves lie in the subspaces V_m, one curve to each subspace V_m; and he gives proofs of several theorems on the relationship of properties of the curves as elements of V_n to their properties as elements of V_m; for example, a necessary and sufficient condition that each curve of a family $\{C_m\}$ be a geodesic in the V_m which contains it is that the principal normals of the curves C_m relative to V_n be normal to the subspaces V_m. The subject of infinitesimal deformations of this laminated manifold is taken up in the second part of the paper, and conditions are given for tensor fields to remain invariant under the deformation, geodesics of V_m to remain such, harmonic vector fields of V_m to remain harmonic, etc.

W. M. Boothby (Evanston, Ill.).

Ishihara, Shigeru. Correction: Fibred Riemannian spaces with isometric parallel fibres (this journal, Vol. 6, pp. 243-252). Tôhoku Math. J. (2) **8** (1956), 333.

The definition of a fibred Riemannian space given in the previous paper [MR **16**, 1052] was insufficient for the validity of the general discussion, and a modified definition is now given.

A. G. Walker (Liverpool).

Halder, Gita; and Behari, Ram. Generalised asymptotic lines in Riemannian manifolds. Ganita **7** (1956), 45–53.

If V_{n+1} and V_n represent respectively a Riemannian manifold of dimension $(n+1)$ and a hypersurface of V_{n+1}, a curve C lying in V_n is called a generalised asymptotic line in V_n if its tangent vector with respect to the hypersurface V_n is self-conjugate as well as conjugate to its first $(n-2)$ principal normals with respect to V_n. Here are some properties of these curves: Given a generalised asymptotic line, its first $(n-1)$ principal normals with respect to V_{n+1} and V_n coincide; and its nth principal normal with respect to V_{n+1} coincides with the normal to the hypersurface V_n. Furthermore, if a generalised asymptotic line passing through a point in V_n is also conjugate to its $(n-1)$th principal normal with respect to V_n, then at least one of the lines of curvature of V_n passing through the same point is also a generalised asymptotic line in V_n.

F. Şemin (Istanbul).

Guglielmino, Francesco. Corrispondenze puntuali fra due varietà a tre dimensioni che conservano le curve aventi nulla la seconda curvatura. Boll. Un. Mat. Ital. (3) **12** (1957), 57–60.

It is known that a mapping between two 3-dimensional Riemann spaces V_3, $\bar{V}_3$ is such that their geodesics correspond if and only if the tensor $\rho^h{}_{ij}$ equal to the difference of their respective Christoffel symbols at corresponding points satisfies $2\rho^j{}_{ij}-\rho^i{}_{ii}=0$ $(j\neq i)$, $\rho^h{}_{ij}=0$ $(j\geqq i,\ h\neq i,\ j)$. The author proves that these same conditions are necessary and sufficient for correspondence between curves of zero second curvature.

A. Fialkow (Brooklyn, N.Y.).

Prvanovitch, Mileva. A note on the union curvature of the curves of a Riemannian space. Math. Student **24** (1956), 209–215 (1957).

A curve of a subspace V_n imbedded in a Riemann space V_m having the property that, at each point, its osculating geodesic surface contains the tangent vector to a curve of a congruence of curves through this point is called a union curve of the subspace V_n relative to the given congruence. In the special case when the congruence is normal to the subspace V_n, union curves are geodesic curves of V_n. The union curvature vector of

any curve C in V_n is the difference between its curvature vector relative to V_n and the curvature vector of the union curve of V_n which is tangent to C. By straightforward calculations, the author obtains an expression for the union curvature of a curve C of V_n and obtains conditions under which this expression reduces to a simpler one given by Springer [Canad. J. Math. **2** (1950), 457–460; MR **12**, 358] and Mishra [Ann. Soc. Sci. Bruxelles. Sér. I. **65** (1951), 109–115; MR **13**, 868]. These conditions are always satisfied if $n=2$. *A. Fialkow.*

Singh, Kamla Devi; and Mishra, R. S. Subspaces of semi-simple group spaces. Tensor (N.S.) **6** (1956), 115–124.

The authors consider subspaces in a compact semi-simple group space. They first calculate the Christoffel symbols, the second fundamental form, curvature of a curve on the subspace, a derived vector of a unit normal, and then discuss curves, orthogonal congruences of curves, curl of a congruence and non-symmetric connexions in the space.

In many of their theorems, they assume that the n generating vectors $h_{(a)}{}^i$ are all tangential to the subspace, but it seems to the reviewer that such a situation cannot happen because $h_{(a)}{}^i$ are n linearly independent vector fields in an n-dimensional manifold. *K. Yano* (Tokyo).

Libermann, Paulette. Automorphismes infinitésimaux d'une structure symplectique. C. R. Acad. Sci. Paris **242** (1956), 1114–1117.

Soit V_p une variété différentiable de classe C^∞, de dimension p. On désigne par K l'anneau des fonctions numériques de classe C^∞ sur V_p, par T le K-module des champs des vecteurs tangents à V_p, par $\hat{T}_q$ le K-module des q-formes différentielles extérieures sur V_p. Pour $X \in T$, l'opérateur transformation infinitésimale est défini par $\theta(X)=di(X)+i(X)d$, où $i(X)$ est le produit intérieur. Donc une forme fermée φ est invariante par X si $\theta(X)\varphi=di(X)\varphi=0$. On envisage les deux cas suivants: 1) φ (désignée par π) est une p-forme, partout de rang p; V_p est alors munie d'une $SL(p, R)$-structure. L'application $\gamma: X \to i(X)\pi$ définit un isomorphisme de T sur $\hat{T}_{p-1}$. La fonction numérique $\partial(X)$ defini par: $\theta(X)\pi=(\partial(X))\pi$ est la divergence de X. 2) V_p est de dimension paire $p=2n$ et φ (designée par Ω) est de degré 2, partout de rang $2n$; V_{2n} est alors symplectique. L'application $\alpha: X \to i(X)\Omega$ définit un isomorphisme de T sur $\hat{T}_1$.

Cela étant, l'auteur démontre les théorèmes suivants.

Sur une variété V_p (resp. V_{2n}) munie d'une $SL(p, R)$-structure (resp. d'une structure symplectique) l'espace M (resp. $\mathscr{L}$) des automorphismes infinitesimaux de la structure est isomorphe à l'espace vectoriel réel $\hat{M}$ (resp. $\hat{\mathscr{L}}$), de dimension infinie, des $(p-1)$-formes (resp. 1-formes) fermées.

L'isomorphisme α (resp. γ) induit sur $\hat{\mathscr{L}}$ (resp. $\hat{M}$) une structure d'algèbre de Lie dont l'espace $\hat{\mathscr{F}}$ (resp. $\hat{F}$) des formes ~ 0 est un idéal et la structure d'algèbre de Lie quotient dans le groupe de cohomologie $\hat{\mathscr{L}}/\hat{\mathscr{F}}$ (resp. $\hat{M}/\hat{F}$) est abélienne.

Sur une variété symplectique l'ensemble K des fonctions numériques de classe C^∞ possède, outre sa structure d'anneau, une structure d'algèbre de Lie (de dimension infinie) définie par la parenthèse de Poisson et dont le centre est le corps des constantes.

L'isomorphisme $\alpha: X \to i(X)\Omega$ applique l'espace M des automorphismes infinitesimaux de la $SL(2n, R)$-structure sur l'espace $\hat{\mathfrak{M}}$ des 1-formes cofermées relativement à Ω.

Sur une variété riemannienne orientable, l'espace M des transformations infinitesimales laissant invariant le volume est transformé dans la dualité $\beta: T \to \hat{T}_1$ par rapport à la métrique en l'espace vectoriel $\hat{\mathfrak{N}}$ des 1-formes cofermées et tout element de $\{\hat{\mathfrak{N}}, \hat{\mathfrak{N}}\}$ est cohomologue à 0, où $\{\varphi, \psi\}=\delta(\varphi \wedge \psi)$.

Sur une variété symplectique compacte V_{2n} $(n>1)$, toute transformation infinitesimale conforme est un automorphisme infinitésimal.

Sur une variété presque kählerienne compacte V_{2n} $(n>1)$, toute transformation infinitésimale conforme de la métrique riemannienne qui conserve la structure presque complexe est une isométrie infinitésimale.

K. Yano (Tokyo).

Cossu, Aldo. Alcune osservazioni sulle connessioni tensoriali. Rend. Mat. e Appl. (5) **13** (1955), 373–390.

Bompiani considered a space with the so-called tensor connection L^{ik}_{rst} [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1** (1946), 478–482; MR **8**, 404] in which the covariant differential of a contravariant tensor ξ^{ik} is defined by the formula $D\xi^{ik}=d\xi^{ik}+L^{ik}_{rst}\xi^{rs}dx^t$. If the functions L^{ik}_{rst} have the form $\delta_r{}^i L^k_{st}+\delta_s{}^k L^i_{rt}$, then the above covariant differential coincides with the usual one with respect to L^k_{st}.

In this paper, the author studies some special classes of the tensor connections which can be deduced from an ordinary vector connection, transformations of the tensor connections, and some tensors constructed from the given tensor connection, especially the curvature tensor of the given tensor connection. *K. Yano* (Tokyo).

Solodovnikov, A. S. Geodesic classes of $V(K)$ spaces. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 33–36. (Russian)

Some results are announced which appear in Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 45–116 [MR **18**, 930], especially sections 11 and 13. *L. W. Green.*

Ichijō, Yoshihiro. On the space with dominant affine connection. J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1956), 37–46.

A dominant affine connection means an affine connection relating m-dimensional affine spaces A_m attached to every point of an n-dimensional space V_n $(m>n)$. By decomposing A_m into two spaces, i.e., the tangent space A_n of V_n and an $(m-n)$-dimensional subspace A_{m-n} of A_m, the fundamental quantities of the space with dominant affine connection are calculated formally in the same way as a subspace of an affinely connected space. There is stated a necessary and sufficient condition for a space with dominant affine connection to be a subspace of an affinely connected space. Also, a geodesic line and asymptotic line in V_n are defined. Finally, the invariant quantity under projective changes of dominant affine connection, which is analogous to the so-called Weyl projective curvature tensor, is calculated. *A. Kawaguchi.*

Haimovici, A. Sur quelques invariants dans les espaces à connexion affine à trois dimensions. Ann. Polon. Math. **3** (1957), 300–303.

Dans quelques notes publiées les dernières années, nous avons étudié les espaces à connexion affine qui admettent, pour un système de deux directions, des invariants au transport parallèle de ces directions. Un de ces invariants

a été l'angle. Dans la note présente, nous nous proposons l'étude d'un problème analogue: il s'agit de trouver les invariants au transport parallèle d'un système de plusieurs directions dans un espace à connexion affine à trois dimensions. Nous avons trouvé effectivement ces invariants; en même temps, nous avons donné une caractérisation géométrique des espaces qui les admettent. (Author's summary.) *P. O. Bell* (Palo Alto, Calif.).

Tevzadze, G. N. On internal Riemannian geometries of the projective space. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 103–126. (Russian)

The paper deals with a special case of the situation considered in the eighth chapter of A. P. Norden, "Affinely connected spaces" [Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 441] and is written under the direction of Norden. *G. Y. Rainich* (Ann Arbor, Mich.).

Gasparini, Ida Cattaneo. Sulle connessioni affini associate a una data connessione lineare. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 119–126.

Dans la terminologie de l'A., qui est aussi celle du rapporteur [voir l'oeuvre analysée ci-dessous], une connexion linéaire est une connexion infinitésimale sur l'espace fibré $E(V_n)$ des repères linéaires d'une variété différentiable V_n, une connexion affine est une connexion infinitésimale sur l'espace fibré $E(V_n)$ des repères affines. Toute connexion affine induit une connexion linéaire. Inversement à toute connexion linéaire est canoniquement associée une connexion affine. L'A. étudie les différentes connexions affines associées à une même connexion linéaire et qui diffèrent les unes des autres par une 1-forme à valeurs vectorielles. La courbure d'une telle connexion affine faite intervenir, outre la courbure de la connexion linéaire induite, sa torsion et la différentielle absolue de la 1-forme dans la connexion linéaire. L'A. définit de plus la différentielle absolue, dans une connexion affine, d'un champ de points des espaces affines tangents; cette différentielle absolue détermine une 1-forme à valeurs vectorielles qui peut être utilisée pour définir des familles intrinsèques de courbes. *A. Lichnerowicz* (Paris).

★**Lichnerowicz, André. Théorie globale des connexions et des groupes d'holonomie.** Edizioni Cremonese, Roma, 1957. xv+282 pp. 4.000 lire.

Over the past years there has been an almost Babylonian confusion of languages in the literature on differential geometry. The field, which at one time was occupied by those who thought in terms of coordinates, components and indices as a natural outgrowth from analytic geometry, found itself invaded by a new zealous generation whose frame of mind was more adapted to intrinsic concepts and notations; to mappings rather than components. Above all, the invaders were dissatisfied with the standards of rigor of what they termed the "classical" differential geometers, whose "theorems" often were true only "in general", and "locally".

The confusion thus arising, with almost as many notations as there were authors, has settled a good bit by now; and the present volume under review makes an admirable, pioneering contribution in book form toward this consolidation. The reader will find himself mainly confronted with the geometrical concepts rather than with numbers representing them in some way or other. Still, when components are handy, the author has not hesitated to use the much-feared "indices", thereby preparing the student for the "classical" literature which has often been given little more than an occasional glance by the members of the "intrinsic" movement. The compromise seems well-balanced, and is more likely to last than either extreme of the scale of notations.

The subject matter has been confined to the theory of connections in fiber bundles (as a general background), extensively applied to linear, affine, Riemannian, almost-complex, Hermitian and Kählerian connections. Much of the classical theories of highly specialized connections has been pruned out, leaving a solid framework of the subjects enumerated, from a central viewpoint. It may seem somewhat regrettable that the theory of motions and Lie derivatives has been completely ignored; but with this exception one can safely say that the choice of topics has been a very fortunate one; that the material presented is 'most likely to survive", and also most capable of significant further developments.

Throughout, ample use has been made of fiber bundle techniques. Thus a tensor field becomes a function or differential form on the principal bundle of frames with values in a vector space; and covariant differentiation is a modification of exterior derivation — very useful in interpreting, for example, the Bianchi identities.

Lie groups and their elementary properties have been assumed known. (Still, the manifold concept is defined from the ground up, while there is no definition of the differential of a function: just one of those controversial points between "classical" and modern methods.) Holonomy groups thus arise without much pain; and their analysis is quite detailed, from the characterization of their Lie algebra to the study of (local) product spaces.

The text is self-contained, apart from the above limitation and the following: tensor algebra, the Whitney imbedding theorem, Hodge's and de Rham's theorems and Yamabe's result on arcwise connected subgroups of a Lie group. — The style is easy-going and contains many ingenious simplifications. — No exercises have been provided.

Despite the generally favorable impression which this book makes on the reviewer, he notes with regret one disadvantage which it has in common with many other texts in the same language. The student who wants to know more about any of the topics touched upon may find himself somewhat lost despite the bibliography of some 58 titles covering a good deal of the subjects discussed (most references to "classical" works have been deleted!). A simple count showed the number of items referred to (once or more) was only some twelve (20%); and even in those cases examples were found when considerable material was borrowed from a paper while only fairly insignificant details were acknowledged.

A chapter-by-chapter survey of contents: I. Differentiable manifolds, vectors, tensors, Riemannian geometry, mappings; exterior differential forms, Pfaffian systems, Frobenius' Theorem; forms with values in a vector space. II. Homotopy of paths, infinitesimal and linear connections. III. Holonomy groups and curvature; local and infinitesimal holonomy groups, reducibility of Riemannian manifolds. IV. Harmonic and covariant constant forms. V. Almost complex, Hermitian and Kählerian structures, harmonic theory. — Bibliography.

A. Nijenhuis (Seattle, Wash.).

Frankel, Theodore. Homology and flows on manifolds. Ann. of Math. (2) **65** (1957), 331–339.

After a short introduction, the author gives in § 1 a brief summary of the vector formulation of de Rham's

theorem. Let M be an n-dimensional compact manifold and let ρ be a positive scalar density on M. Let T_p be the linear space of all p-tensors defined on all of M. The divergence $\partial\mathfrak{w}$ of a p-tensor $\mathfrak{w}=w^{i_1i_2\cdots i_p}$ is defined to be $\partial\mathfrak{w}=\rho^{-1}\partial(\rho w^{ii_2\cdots i_p})/\partial x^i$. We define the linear space of divergence-free p-tensors by $V_p=\{\mathfrak{w}\in T_p\colon \partial\mathfrak{w}=0\}$, and the linear space of p-tensor divergences by $D_p=\{\mathfrak{w}\in T_p\colon \mathfrak{w}=\partial\mathfrak{w}' \text{ for some } \mathfrak{w}'\in T_{p+1}\}$. If we form $W_p=V_p/D_p$, then the vector formulation of de Rham's theorem states that $W_p\approx H_p$, where H_p is the pth homology group of M. If we define an $(n-p)$-form $\mathfrak{w}^*$ by

$$\mathfrak{w}^*=\rho w^{i_1i_2\cdots i_p}\varepsilon_{i_1i_2\cdots i_pi_{p+1}\cdots i_n}dx^{i_{p+1}}\wedge\cdots\wedge dx^{i_n},$$

then we have $d(\mathfrak{w}^*)=(\partial\mathfrak{w})^*$. Thus, the operator $*$ induces an isomorphism between W_p and $(n-p)$th de Rham group R_{n-p}. We denote by $[\mathfrak{w}^*]$ the $(n-p)$-dimensional cohomology class represented by $\mathfrak{w}^*$. We denote also by M the fundamental n-cycle on the manifold, and we can define $[\mathfrak{w}]=[\mathfrak{w}^*]\cap M$, where $\cap$ denotes the cap product. Hence $[\mathfrak{w}]$ is a p-dimensional homology class. A vector $\mathfrak{w}\in T_1$ is called a flow if $\partial\mathfrak{w}=0$. A Killing vector is a flow, since it satisfies $\nabla_jv_i+\nabla_iv_j=0$, and consequently $\nabla_iv^i=0$.

In § 2, the author proves the following theorem. Every closed trajectory of an isometric flow $\mathfrak{v}$ lies in a homology class which is some non-zero multiple of the class $[\mathfrak{v}]$. If the field $\mathfrak{v}$ vanishes at some point, then $[\mathfrak{v}]=0$.

In § 3, the author gives an example which shows that we cannot expect the above theorem to be valid if the flow is not isometric.

In § 4, the author assumes that M is any coset space G/g of a compact Lie group G modulo a closed subgroup g. Then M is a compact homogeneous space acted on transitively by G, and has g for a stability group. M admits a Riemannian metric invariant under G. The volume element $d\tau$ is then also invariant under G.

For $s\in G$, let $T_s\colon M\to M$ denote the action of the group on M; briefly, $T_s(m)=sm$ for $m\in M$. Let dT_s and δT_s denote mappings of tensors and forms induced by T_s. A p-tensor $\mathfrak{w}$ (p-form α) defined on all of M is said to be invariant under G if $dT_s\mathfrak{w}_m=\mathfrak{w}_{sm}$ ($\delta T_s\alpha_{sm}=\alpha_m$) for all $s\in G$, $m\in M$. If $d\mu$ is the normalized invariant volume element on G and α is a p-form on M, then $(I\alpha)_m=\int_G \delta T_s(\alpha_{sm})\,d\mu(s)$ is an invariant p-form on M; and Cartan's theorem states that if α is closed then $I\alpha$ is closed and defines the same cohomology class as α. For the vector formulation of this result we define, for each p-tensor $\mathfrak{w}$, the invariant tensor $(I\mathfrak{w})_m=\int_G dT_s(\mathfrak{w}_{s^{-1}m})\,d\mu(s)$. Then the author proves the following theorems. If $\partial\mathfrak{w}=0$ with respect to the invariant density on M, then $\partial(I\mathfrak{w})=0$ and $I\mathfrak{w}$ defines the same homology class as $\mathfrak{w}$. Let a one-parameter subgroup A of G act on M. If any closed one-dimensional orbits on M result from this action, then those orbits lie in dependent homology classes. If A has a fixed point, then all closed orbits bound.

After proving a lemma on exterior products, the author gives in § 6 a proof of Weyl's theorem. If G is compact and semi-simple, then the first Betti number is zero.

In the last section the author proves the theorem: If $\mathfrak{v}$ is a divergence-free vector field on M there is a vector field $\mathfrak{w}$, tangent to a flow generated by a one-parameter subgroup of G, such that $\mathfrak{w}$ is homologous to $\mathfrak{v}$.

K. Yano (Tokyo).

Kostant, Bertram. On differential geometry and homogeneous spaces. I, II. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 258–261, 354–357.

In a previous paper [Trans. Amer. Math. Soc. **80** (1955), 528–542; MR **18**, 930] the author proved: Let M be a compact Riemannian manifold, X an infinitesimal motion on M, and a_X the endomorphism of the tangent space V_o at an arbitrary point $o\in M$ defined by $a_Xv=-\nabla_vX$ for any $v\in V_o$, where ∇_v denotes the covariant differentiation with respect to v. Then $a_X\in\mathfrak{s}_o$, the Lie algebra of the restricted homogeneous holonomy group at o. He also proved in the same paper the following theorem: If, in addition, M is homogeneous, say $M=G/K$, where G is a compact connected Lie group acting transitively as motions on M, then the a_X's actually generate $\mathfrak{s}_o$, as a Lie algebra, where X runs through the infinitesimal motions of M arising from the Lie algebra $\mathfrak{g}$ of G.

Now let G be a compact connected Lie group, K a compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras. For $X\in\mathfrak{g}$, let $X^*=\pi(X)$ be the vector field on $M=G/K$ whose value at an arbitrary point $p\in M$ is given by the curve $\exp tX\cdot p$, $t\geqq 0$, with initial point $p\in M$. Let $\mathfrak{p}$ be a complement for $\mathfrak{k}$ in $\mathfrak{g}$ such that $\operatorname{ad} u$ leaves $\mathfrak{p}$ invariant for any $u\in K$. Let $u\to\sigma u$ and $X\to\sigma X$, $u\in K$, $X\in\mathfrak{k}$, designate the respective representations of K and $\mathfrak{k}$ on $\mathfrak{p}$ obtained by restricting $\operatorname{ad} u$ and $\operatorname{ad} X$ to $\mathfrak{p}$. Finally, let $P\colon\mathfrak{g}\to\mathfrak{p}$ be the projection of $\mathfrak{g}$ on $\mathfrak{p}$ which vanishes on $\mathfrak{k}$. The author first proves the following lemma. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a decomposition of $\mathfrak{g}$. A necessary and sufficient condition that the corresponding canonical affine connection on $G/K=M$ arise from a Riemannian metric (necessarily invariant) is that the Lie algebra $\mathfrak{h}_o$ of endomorphisms of $\mathfrak{p}$ generated by all $P\operatorname{ad}X\,P$, $X\in\mathfrak{g}$ be a compact Lie algebra. The author calls $\mathfrak{p}$ a natural complement, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a natural decomposition and the corresponding canonical connection a natural connection if this connection is Riemannian. Applying his previous results, he obtains the following theorem. Let $\mathfrak{p}$ be a natural complement to $\mathfrak{k}$. Let G/K be given the corresponding cannonical affine connection; then $\mathfrak{h}_o=\mathfrak{s}_o$, the holonomy algebra at o. He then states the second lemma: The transformation $\exp sa_X$ of V_o may be realized by transporting V_o from o to $\exp sX\cdot o$ by parallelism along the curve $\exp tX\cdot o$, $0\leqq t\leqq s$, and returning V_o to o by the differential of the motion $\exp(-sX)$. Using this lemma, he proves then the theorem: The full homogeneous holonomy group ψ_o of G/K is given by $\psi_o=\sigma(K)\exp\mathfrak{h}_o$. Also, the theorem: Let $h(t)$ be any geodesic on G/K which intersects itself; then $h(t)$ is a closed geodesic.

In the second paper, the author proves first the following theorem. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{k}\cap\mathfrak{p}=0$, $[\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{p}$. Let a bilinear form B be strictly invariant on $\mathfrak{p}$. Let $\mathfrak{g}(\mathfrak{p})=\mathfrak{p}+[\mathfrak{p},\mathfrak{p}]$, $\mathfrak{k}_1=\mathfrak{g}(\mathfrak{p})\cap\mathfrak{k}$, so that the ideal $\mathfrak{g}(\mathfrak{p})=\mathfrak{k}_1+\mathfrak{p}$. There exists one and only one invariant bilinear form B^* on $\mathfrak{g}(\mathfrak{p})$ extending B and such that $B^*(\mathfrak{k}_1,\mathfrak{p})=0$. Moreover, B^* is nonsingular on $\mathfrak{g}(\mathfrak{p})$ and hence on $\mathfrak{k}_1$. He then states several consequences of this theorem. The author closes the paper by proving a generalisation of a theorem of Cartan on symmetric spaces: Let ψ_o be the holonomy group at o. A subspace $\mathfrak{p}_1\subset\mathfrak{p}$ is invariant under ψ_o if and only if it is of the form $\mathfrak{p}_1=P\mathfrak{g}_1$, where $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ such that $\mathfrak{g}_1=(1-P)\mathfrak{g}_1+P\mathfrak{g}_1$. Moreover, if $\sum_{i=1}^{m}\mathfrak{p}_i=\mathfrak{p}$ is a direct sum decomposition of $\mathfrak{p}$ into invariant subspaces of ψ_o (irreducible or not), then (1) $\mathfrak{g}_i=\mathfrak{p}_i+[\mathfrak{p}_i,\mathfrak{p}_i]$ ($i=1,2,\cdots,m$) are ideals in $\mathfrak{g}$; (2) $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$ is a natural decomposition of $\mathfrak{g}_i$; and (3) $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\oplus\cdots\oplus\mathfrak{g}_m$ and $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2\oplus\cdots\oplus\mathfrak{k}_m$, where $\mathfrak{k}_i=\mathfrak{g}_i\cap\mathfrak{k}$. Finally, if $\mathfrak{c}$ is the centre of $\mathfrak{g}$, then $\mathfrak{p}\cap\mathfrak{c}$ is the space of fixed vectors for ψ_o.

K. Yano (Tokyo).

Vranceanu, G. Sugli spazi omogenei a connessione affine. Rend. Circ. Mat. Palermo (2) **5** (1956), 288–296 (1957).

An affine manifold $A_n(x^1, x^2, \cdots, x^n)$ is called homogeneous if it admits a transitive group of transformations in itself. It is shown that, if the group is a G_{n+r}, then the torsion and curvature tensor are zero if they are zero at one point (taken as origin). If the A_n is non-euclidean, then the group cannot contain the homothetic group $x'^i=ax^i$. Then, starting with the general space $X_{n+r}(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+r})$ associated with G_{n+r}, a new demonstration is given of a theorem of E. Cartan [Bull. Soc. Math. France **54** (1926), 214–264, esp. p. 225= Oeuvres complètes, partie I, v. 2, Gauthier-Villars, Paris, 1952, pp. 587–637, esp. p. 598; MR **14**, 343] in the form that if the equations of structure of a homogeneous A_n admit a solution $ds_k^h=\gamma_{kl}^h ds^l+a_{k\beta}^h ds^\beta (h, k, l=1, \cdots, n; \alpha, \beta, \cdots=n+1, \cdots, n+r)$ with γ_{kl}^h zero, then it is symmetric [see the author's "Leçons de géometrie différentielle, v. I, Bucarest, 1947, p. 294; MR **9**, 532]. This leads also to a theorem by P. K. Raševskiĭ [Trudy Sem. Vektor. Tenzor. Anal. **8** (1950), 82–92; MR **12**, 534] which states that if the constants $c_{l\alpha}^\beta$ of the G_{n+r} are zero, then there exists a general symmetric A_n which admits G_{n+r} as group of motions. An example of a symmetric group ($c_{kl}^h=c_{l\beta}^\alpha=0$) which is the group of motions of a non-symmetric space is the case of the A_n with maximum group G_{n^2} [Vranceanu, Acad. R. P. Romîne, Bul. Şti Secţ. Şti. Mat. Fiz. **6** (1954), 49–59; MR **16**, 1151]. *D. J. Struik.*

Willmore, T. J. Parallel distributions on manifolds. Proc. London Math. Soc. (3) **6** (1956), 191–204.

This article contains proofs of theorems reported by the author at the Amsterdam congress in 1954. Closely related results appear in Walker, Quart. J. Math., Oxford Ser. (2) **6** (1955), 301–308 [MR **19**, 312]. We mention certain results on Riemannian connections. It is shown that for $n=2$ and $n\geq 5$ there exist compact orientable differentiable manifolds on which no continuous vector field is parallel relative to a positive definite Riemannian metric. The proof depends on Betti numbers; it is shown that the Betti numbers of a compact orientable manifold allowing such a parallel vector field satisfy $\sum_{s=0}^{n}(-1)^s B_s=0$ (vanishing of the Euler–Poincaré characteristic) and $B_r\geq\sum_{s=0}^{r-1}(-1)^{s+1}B_{r-s}+(-1)^{r+1}$; $r=1, 2, \cdots, [n/2]$. Certain non-existence theorems for distributions of m-planes, $m>1$, are proved by using theorems on tangent bundles of the n-sphere [Steenrod, The topology of fibre bundles, Princeton, 1951; MR **12**, 522]. *V. V. Ryžkov* (RŽMat **1957**, no. 2642).

Willmore, T. J. Systems of parallel distributions. J. London Math. Soc. **32** (1957), 153–156.

Fibre bundle theory is used to give a short proof of the theorem, first proved by the reviewer [Quart. J. Math. Oxford Ser. (2) **6** (1955), 301–308], that for any system of disjoint distributions over a C^∞ manifold there exists a connexion with respect to which each distribution is parallel, this connexion being torsion-free when the system is integrable. *A. G. Walker* (Liverpool).

de Mira Fernandes, A. Sugli pseudo-estensori jacobiani. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 249–264.

The author [same Revista **2** (1952), 361–380; MR **15**, 900] has defined pseudo-extensors in relation to the extensors of Craig. Craig and Guy [Amer. J. Math. **42** (1950), 229–246; MR **11**, 543] had defined jacobian extensors, these being generalizations of tensor densities. In this paper the author proves that it is not possible to generalize the definition of pseudo-extensors so that it could be applied to jacobian extensors. *J. A. Schouten* (Epe).

Hölder, E. Über die auf Extremalintegrale gegründeten metrischen Räume. Schr. Forschungsinst. Math. **1** (1957), 178–193.

Cet article reproduit (avec toutefois une modification à la fin) une conférence donnée à Berlin en Octobre 1954 à l'occasion du centième anniversaire de la "Probevorlesung" de Riemann. Suivant E. Cartan l'auteur esquisse l'étude géométrique des espaces variationnels envisagés comme espaces à connection euclidienne. S'il s'agit d'intégrales de dimension $\mu=1$ (cas des espaces de Finsler) dans l'espace R_n, à tout élément linéaire de contact (x, p) est attachée une métrique riemannienne $ds^2=g_{ij}(x, p)dx^i dx^j$, à deux éléments linéaires de contact voisins (x, p), $(x+dx, p+dp)$ et un champ de vecteurs $X(x)$ une différentielle absolue $DX^i=dX^i+X^k(C_{k\cdot h}^{\cdot i}dp^h+\Gamma_{k\cdot h}^{\cdot i}dx^h)$. Il est procédé de manière duale dans le cas d'intégrales de surface de dimension $\mu=n-1$ (espaces métriques fondés sur la notion d'aire d'E. Cartan). Les coefficients g_{ij}, $C_{k\cdot h}^{\cdot i}$, $\Gamma_{k\cdot h}^{\cdot i}$ sont reliés à l'intégrant $L(x, dx)$ du problème variationnel par les conditions intrinsèques d'E. Cartan, **A**—**E** pour $\mu=1$, **A***—**E*** pour $\mu=n-1$, qui les définissent de manière unique sauf dans le cas de l'espace singulier d'E. Cartan. **A**—**E** d'une part, **A***—**E*** d'autre part ne sont pas duales. Il en résulte des difficultés de transfert au cas de μ quelconque $=1, 2, \cdots, n-1$, étudié par Kawaguchi et ses élèves au prix de calculs d'une extrême complication. Plus simple et plus intuitive est la théorie de H. Boerner [Math. Z. **46** (1940), 720–742; MR **2**, 225] et W. Velte [Mitt. Math. Sem. Giessen no. **45** (1953); Math. Z. **60** (1954), 367–383; MR **16**, 492] qui, suivant Carathéodory, reste plus étroitement en contact avec le Calcul des Variations. Celle-ci est exposée avec plus de détails. L'article se termine sur une remarque où sont proposées pour μ quelconque des conditions équivalentes à **A**—**E** pour $\mu=1$ et à **A***—**E*** pour $\mu=n-1$. *Chr. Pauc.*

See also: Harmonic Functions, Convex Functions: Friedrichs; Morrey and Eells; Morrey. **Partial Differential Equations:** Schwartz. **Geometries, Euclidean and Other:** Lippmann. **Differential Geometry:** Mihăileanu; Grigor'ev; Picasso. **Complex Manifolds:** Legrand. **Mechanics of Particles and Systems:** Freudenthal.

Complex Manifolds

Legrand, Gilles. Structures presque hermitiennes au sens large. C. R. Acad. Sci. Paris **243** (1956), 1392–1395.

Soit V_{2n} une variété différentiable de classe C^∞, et $T_x{}^C$ le complexifié de l'espace vectoriel tangent au point $x\in V_{2n}$. On suppose que V_{2n} est munie d'une structure de presque-produit complexe définie en chaque point x par un opérateur linéaire F sur $T_x{}^C$ tel que $F_j{}^i F_i{}^h=\lambda^2\delta_j^h$. Une métrique complexe est définie sur V_{2n} par la donnée d'un tenseur symétrique g_{ji} dont les composantes dans un système de coordonnées locales (x^i) sont des fonctions à valeurs complexes des x^i, de classe C^∞, à déterminant partout $\neq 0$. On dit que la métrique est hermitienne par rapport à F si l'on a $F_j{}^i F_i{}^k g_{lk}=-\lambda^2 g_{jl}$ et que la structure de presque-produit complexe F et une telle metrique g_{ji} définissent sur V_{2n} une structure presque hermitienne au

sens large. Si $\lambda=i$ et la métrique est réelle et définie positive, on retrouve la notion habituelle de structure presque hermitienne.

L'auteur obtient la connexion de Lichnerowicz [Arch. Math. **5** (1954), 207–215; MR **16**, 520] et celle de Chern-Libermann [Chern, Ann. of Math. (2) **47** (1946), 85–121; MR **7**, 470; Libermann, Ann. Mat. Pura Appl. (4) **36** (1954), 27–120; MR **16**, 520] dans un espace à structure presque hermitienne au sens large. Dans la dernière partie de la note, il généralise les propriétés des groupes d'holonomie des connexions presque hermitiennes au sens de Lichnerowicz. *K. Yano* (Tokyo).

Guggenheimer, H. Opérateurs différentiels et suites exactes sur une variété Kählérienne. Tôhoku Math. J. (2) **8** (1956), 308–323.

Sur une variété kaehlérienne M de coordonnées locales $(dz_i, d\bar{z}_i)$, de forme fondamentale Ω, on appelle type d'une forme homogène $\varphi_{(s)}{}^r$, de degré total r, son degré s par rapport aux $d\bar{z}_i$; on pose $L\varphi^r=\varphi^r\wedge\Omega$, $\Lambda\varphi^r=(-1)^{r*}L*\varphi^r$; $\varphi_k{}^r$ est dite de classe k si $\varphi_k{}^r=L^k\varphi_0{}^{r-2k}$ et $\Lambda\varphi_k{}^r=0$. Si V est une sous-variété régulièrement et analytiquement plongée dans M, V étant sans singularité, la restriction d'une forme $\varphi_k{}^r=\Omega^k\wedge\varphi_0{}^{r-2k}$ vérifie $R\varphi_k{}^r=\omega^k\wedge R\varphi_0{}^{r-2k}$; elle ne diminue pas la classe. Etant donnée une forme φ sur V, supposée paracompacte, on construit inversement une extension $E\varphi$ de φ, $E\varphi$ étant une forme définie sur M avec $\varphi=R(E\varphi)$. Cette extension conserve la classe. Une modification analytique de M en une variété kaehlérienne M' (M et M' étant supposées compactes et la modification étant analytique biunivoque de $M-V$ en $M'-V'$, où V et V' sont des sous-variétés analytiques sans singularité) induit un isomorphisme $H_{k,s}{}^r(M, V)\cong H_{k,s}{}^r(M', V')$ auquel on fait appel pour indiquer des propriétés des genres géométriques ρ_{a0}: on a $\rho_{a0}(M)=\rho_{a0}(M')$ et $\rho_{a0}(V)=\rho_{a0}(V')$. D'une discussion selon les valeurs des dimensions complexes v, v' de V, V' comparées à la dimension complexe n de M, on déduit en particulier que pour $v\neq v'$, il est impossible qu'on ait $v+v'<n-1$.

P. Lelong (Paris).

Lichnerowicz, André. Sur les automorphismes de certaines variétés kähleriennes. Bull. Soc. Math. Belg. **8** (1956), 3–14.

Développant des résultats établis antérieurement par lui [C. R. Acad. Sci. Paris **239** (1954), 1344–1346; **241** (1955), 726–729; MR **17**, 531, 660], l'A. prouve que, sur une variété Kählérienne irréductible, à courbure de Ricci non nulle, le plus grand groupe connexe d'isométries est aussi le plus grand groupe connexe G d'automorphismes. Dans le cas des variétés Kählériennes compactes qui sont en même temps espaces d'Einstein, il y a isomorphisme entre l'algèbre de Lie de G et une algèbre définie sur l'espace des solutions de $\Delta\varphi=\lambda_0\varphi$, où $\lambda_0=R/n$. Application aux espaces homogènes Kählériens compacts.

J. Lelong (Paris).

★ **Remmert, Reinhold. Über stetige und eigentliche Modifikationen komplexer Räume.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 17 pp.

Die Arbeit gibt einen Bericht über die von dem Verfasser und Grauert gewonnenen Resultate zur Theorie der Modifikationen. Diese basiert auf dem von Behnke und Stein eingeführten Begriffe des komplexen Raumes, dessen Modifikationen eingehend analysiert werden. Dabei erweisen sich die stetigen und eigentlichen Modifikationen als grundlegend. Als Beispiel einer eigentlichen Modifikation wird der Hopfsche σ-Prozess genannt. Ein komplexer Raum, der nicht aus einem anderen komplexen Raum durch eine echte eigentliche Modifikation entstehen kann, wird ein komplexer Urraum genannt. Es werden Klassen solcher Urräume angegeben. Zu den Urräumen gehören die komplexen projektiven Räume $P^n(C)$. [Zur ausführlichen Darstellung s. Grauert und Remmert, Math. Ann. **129** (1955), 274–296; MR **17**, 81.]

F. Sommer (Münster).

★ **Stein, Karl. Analytische Abbildungen allgemeiner analytischer Räume.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 9 pp.

Remmert bewies den Satz: Ist $\tau: X\to Y$ eine holomorphe Abbildung eines kompakten komplexen Raumes X in einen beliebigen komplexen Raum Y, so ist $\tau(X)$ eine analytische Menge in Y. Versteht man unter dem Range r der Abbildung τ die minimale Codimension der Faser von τ, so gilt: Y ist eine r-dimensionale analytische Menge. Aus diesem Satz ergeben sich Sätze von Chow und Thimm als einfache Folgerungen. Satz von Chow: Ist X ein zusammenhängender kompakter n-dimensionaler komplexer Raum, so ist der Körper $K(X)$ der auf X meromorphen Funktionen isomorph einer algebraischen Erweiterung eines Körpers in höchstens n Unbestimmten über dem Körper C der komplexen Zahlen. Satz von Thimm: In kompakten komplexen Räumen sind analytisch abhängige Funktionen stets algebraisch abhängig. [Eine ausführliche Darstellung findet man in Remmert, Math. Ann. **132** (1956), 277–288; MR **19**, 171.] *F. Sommer* (Münster).

Grauert, Hans; und Remmert, Reinhold. Konvexität in der komplexen Analysis. Nicht-holomorph-konvexe Holomorphiegebiete und Anwendungen auf die Abbildungstheorie. Comment. Math. Helv. **31** (1956), 152–160, 161–183.

1. Par domaine de Riemann $\mathfrak{G}=(G, \phi)$ au dessus de C^n, on entend un espace topologique G dont tout point $x\in G$ possède un voisinage U_x projeté par ϕ sur C^n, U_x étant un revêtement analytique ramifié, au sens usuel, de $\phi(U_x)$. Si ϕ est localement biunivoque, $\mathfrak{G}$ est dit non ramifié. On définit: 1) une partie E d'un espace analytique X est dite mince d'ordre k si tout point $x\in X$ a un voisinage U_x dans lequel $E\cap U_x$ appartient à un ensemble analytique de dimension complexe au plus $n-k$; 2) un point r de la frontière ∂G de G est dit inessentiel s'il possède un voisinage U_r dans lequel G est non ramifié, $U_r\cap\partial G$ étant mince d'ordre 1. La pseudo-convexité d'un domaine $\mathfrak{G}$ non ramifié est définie à partir du théorème de continuité; son étude se fait comme pour les domaines de C^n [cf. P. Lelong, J. Analyse Math. **2** (1952), 178–208; MR **14**, 971] et n'apporte pas de fait nouveau, les énoncés obtenus traduisant le fait que G est convexe par rapport à la classe des fonctions plurisousharmoniques dans G. Un théorème de K. Oka [Jap. J. Math. **23** (1953), 97–155; MR **17**, 82] entraîne d'autre part: un domaine $\mathfrak{G}$ non ramifié n'est domaine d'holomorphie que s'il est holomorphe-convexe.

2. On dira qu'un ensemble fermé $E\subset\partial G$ a la propriété (A) si E est formé de points frontières essentiels et si, de plus, tout $x\in E$ a un voisinage U_x tel qu'il existe une fonction $f\neq 0$, holomorphe dans $U_x\cap G$, tout point $y\in(E\cap U_x)$ étant limite d'une suite y_ν pour laquelle $\lim_{\nu\to\infty} f(y_\nu)=0$. On établit: si $E\subset\partial G$ a la propriété (A) et si $\mathfrak{G}$ est non ramifié, pseudo-convexe en tout point de $\partial G-E$, $\mathfrak{G}$ est

pseudo-convexe; $\mathfrak{G}$ est alors domaine d'holomorphie. La démonstration utilise, par l'application ϕ, la propriété de la distance $\delta_p(x)$ de $x \in G$ à ∂G, comptée parallèlement à un sous-espace L^p de C^n: $-\log \delta_p(x)$ est plurisousharmonique. On s'appuie également sur le lemme: $\mathfrak{G}$ non ramifié est pseudo-convexe en tout point de ∂G, dès qu'il l'est en tout point de ∂G non projeté sur e, e étant un ensemble analytique.

3. On reprend l'étude d'un ensemble analytique X^{k+1} déjà utilisé par les auteurs [Math. Z. **67** (1957), 103–128; MR **19**, 317] et défini par les équations $z_1/w_1=z_2/w_2=\cdots=z_k/w_k$ dans C^{2k}; X^{k+1} est représenté sur $C^{k+1}(v_0, v_1, \cdots, v_k)$ par l'application ϕ: $v_0=z_1+z_2+\cdots+z_k$, $v_j=w_j+\alpha_j z_j$, $1\leqq j\leqq k$, $|\alpha_j|=1$, $\alpha_j\neq\alpha_i$ for $i\neq j$.

On pose $w_j=sz_j$ et on considère $\mathfrak{G}=(G, \varphi)$, G étant le domaine des points de $X^{k+1}-0$ pour lesquels $|s|<d<1$; G est isomorphe au produit Y de $|s|<d$ par C^k-0. $\mathfrak{G}$ muni de la projection φ sur C^{k+1} est un domaine ramifié au dessus de C^{k+1}, et possède un unique point frontière r_0 projeté à l'origine de C^{k+1}. On établit: 1) $\mathfrak{G}$ est un domaine d'holomorphie; 2) $\mathfrak{G}$ est pseudo-convexe en tout point frontière autre que r_0; 3) $\mathfrak{G}$ n'est pas convexe par rapport aux fonctions holomorphes dans $\mathfrak{G}$. Preuve est ainsi faite que, pour $n\geqq 3$, il existe des domaines ramifiés, à un nombre fini de feuillets, au dessus de C^n qui sont des domaines d'holomorphie et ne sont ni pseudo-convexes ni holomorphes-convexes. D'autre part $r_0 \in \partial G$ est un ensemble qui a la propriété (A); l'énoncé établi pour les domaines $\mathfrak{G}$ non ramifiés ne s'étend donc pas aux domaines ramifiés. Le travail contient encore le théorème suivant: si X^n est un espace analytique, si $E \subset X^n$ est mince d'ordre 2, si π est une application de X^n dans C^n dont le déterminant ne s'annule pas hors de $E \cup K$, K étant l'ensemble des points non uniformisables de X^n, alors le déterminant de π ne s'annule pas; X^n est alors une variété.

P. Lelong (Paris).

See also: Integral and Integrodifferential Equations: Kohn and Spencer. **Algebraic Topology:** Guggenheimer. **Manifolds, Connections:** Lichnerowicz.

Algebraic Geometry

Barlotti, Adriano. Una limitazione superiore per il numero di punti appartenenti a una k-calotta $C(k, 0)$ di uno spazio lineare finito. Boll. Un. Mat. Ital. (3) **12** (1957), 67–70.

Let $S_{n,q}$ be a projective space of dimension n over a Galois field of order q, and consider a set C of k points of $S_{n,q}$ such that no three points of C are collinear. It is known that $k\leqq q+1$ if $n=2$, the equality sign holding here if, and only if, C consists of the points of a conic [B. Segre, Canad. J. Math. **7** (1955), 414–416; MR **17**, 72]; moreover $k\leqq q^2+1$ if $n=3$ and q is odd, the equality sign holding here if, and only if, C consists of the points of a non-ruled quadric [A. Barlotti, Boll. Un. Mat. Ital. (3) **10** (1955), 498–506; MR **17**, 776]. Further, in the case when $n>3$ and $q>2$, it was shown that $k<q^{n-1}+1$ [G. Tallini, Ann. Mat. Pura Appl. (4) **42** (1956), 119–164; MR **19**, 55]; the present paper improves on the last inequality (also in the case when q is even), by showing, e.g., that $k\leqq q^{n-1}-(q-5)(q^{n-3}-1)/(q-1)+1$ if $n>3$ and q is odd and $\geqq 7$.

B. Segre (Rome).

Herszberg, J. Algebraic characterization of types of unodes of surfaces in S_3. J. London Math. Soc. **32** (1957), 187–198.

Isolated double points of algebraic surfaces which have no double lines in any neighbourhood were classified by the reviewer [Proc. Cambridge Philos. Soc. **30** (1934), 453–459, 460–465]. They are the conic node and binodes B_s ($s\geqq 2$, B_2 denoting the conic node), the ordinary unodes U_s ($s\geqq 6$), and three types of exceptional unode U_8^*, U_9^*, U_{10}^*. The equation of a surface with B_s at the origin of affine coordinates in S_3 had been shown long before by H. P. Hudson [Proc. 5th Internat. Congress Math., Cambridge, Eng., 1912, Cambridge, 1913, pp. 118–121] to be of the form $H\cdot K+f_s=0$, where H, K are polynomials containing linear terms which are linearly independent, f_s is a polynomial whose lowest terms are of order s, and the factorisations cannot be carried further (i.e., if the coordinate system is so chosen that the linear terms in H, K are x, y, f_s has a non vanishing term in z^s).

The present paper obtains analogous standard forms for a surface with a unode at the origin, namely

$$U_s\colon (x+u_2+\cdots+u_{s-5})^2 +(y+\phi_2+\cdots+\phi_{s-6})^2(z+\psi_2+\cdots+\psi_{s-6}) +w_{s-3}+\cdots+w_n=0 (w_{s-3}(0,0,1)\neq 0),$$

$$U_8^*\colon (x+u_2+u_3)^2+(y+\phi_2)^3+z^4+w_5+\cdots+w_n=0,$$

$$U_9^*\colon (x+u_2+u_3)^2+(y+\phi_2)^3+yz^3+w_5+\cdots+w_n=0,$$

$$U_{10}^*\colon (x+u_2+u_3+u_4)^2+(y+\phi_2+\phi_3)^3+z^5+w_6+\cdots +w_n=0,$$

where u_i, ϕ_i, ψ_i, w_i denote forms of degree i.

P. Du Val (London).

Herszberg, J. Classification of isolated double points of rank zero on primals in S_n. J. London Math. Soc. **32** (1957), 198–203.

This paper is a classification of isolated double points of a primal in S_n, which have no double locus of dimension >0 in any neighbourhood. K_2 denotes the tangent cone, given by the vanishing of the quadratic terms only in the equations of the primal (in an affine coordinate system with origin at the double point) and K_3 the cone similarly given by the vanishing of the cubic terms. It is first shown that a point in the first neighbourhood of the origin is double on the primal if and only if it is common to K_3 and the vertex of K_2, whence if the vertex of K_2 is k-dimensional the locus of double points in the neighbourhood is at least $(k-2)$-dimensional (when it is cubic) but may be $(k-1)$-dimensional and linear if the vertex of K_2 lies on K_3. If there are no double loci of dimension >0 in any neighbourhood we have the following possibilities: (1) $(k=0)$ $N_2^{(0)}$ — no singularity in the neighbourhood. (2) $(k=1)$ $N_s^{(1)}$ — $(s\geqq 3)$ with equation

$$X_1^2+\cdots+X_{n-1}^2+x_n^3+f_{s+1}+\cdots+f_m=0,$$

where X_i is a polynomial of degree $s-1$ with x_i as sole linear term, and f_i a form of degree i. $N_3^{(1)}$ has no singularity in its neighbourhood, $N_4^{(1)}$ has an $N_2^{(0)}$, and $N_s^{(1)}$ ($s\geqq 5$) an $N_{s-2}^{(1)}$, and so forth along the curve $X_1=\cdots=X_{n-1}=0$. (This is analogous to a binode B_s on a surface.) (3) $(k=2)$ Ordinary $N_s^{(2)}$ ($s\geqq 6$), and exceptional ${}^*N_8^{(2)}$, ${}^*N_9^{(2)}$, ${}^*N_{10}^{(2)}$ — these are analogous to the ordinary and exceptional unodes of a surface, and have exactly analogous families of double points in their neighbourhoods. The equations are like those of the corresponding unodes [see preceding review] except that the first term $(x+\cdots)^2$ in each is replaced by a sum of $n-2$ terms $(x_1+\cdots)^2+\cdots+(x_{n-2}+\cdots)^2$ and y, z by x_{n-1}, x_n.

P. Du Val (London).

Boughon, Pierre; Nathan, Jacqueline; et Samuel, Pierre. Courbes planes en caractéristique 2. Bull. Soc. Math. France **83** (1955), 275–278.

Verfasser untersuchen irreduzible ebene Kurven der Ordnung n über einem Grundkörper der Charakteristik 2 und heben deren Besonderheiten gegenüber dem klassischen Fall hervor. Dabei ergibt sich vor allem, dass der Schnittpunkt zweier benachbarter Tangenten ("point caracteristique") im allgemeinen (Verfasser behaupten "immer") vom Berührungspunkt verschieden ist und dass die Klasse der Kurve höchstens $\frac{1}{2}n(n-1)$ erreicht. So bilden z.B. die Tangenten eines Kegelschnittes ein Büschel. *W. Gröbner* (Zbl **65** (1957), 364).

Segre, Beniamino. Intorno alla geometria sopra un campo di caratteristica due. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 97–123. (Turkish summary)

Etude de propiétés géométriques amusantes de quelques courbes algébriques élémentaires sur un corps de caractéristique 2. Toutes les tangentes à une conique passent par un même point. Le point caractéristique de la tangente à une courbe algébrique plane C est toujours distinct du point de contact P (c'est le tangentiel de P lorsque C est la cubique "équianharmonique" $x^3+y^3+z^3=0$). La classe d'une courbe plane générale d'ordre n est $\frac{1}{2}n(n-1)$. Classification des cubiques planes; étude de leurs points d'inflexion. Certains quartiques planes ont une infinité de droites bitangentes, et de coniques quadritangentes. Etude de la surface engendrée par les tangentes à une cubique gauche. Classification des quartiques gauches elliptiques. Le nombre des coniques irréductibles d'un plan qui sont définies sur le corps à q éléments est q^5-q^3. Une partie des résultats sur les courbes planes a été antérieurement trouvée par P. Boughon, J. Nathan et le rapporteur [voir l'analyse ci-dessus]. *P. Samuel.*

Nagata, Masayoshi. A treatise on the 14-th problem of Hilbert. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 57–70.

[Pour la terminologie, voir Nagata, Amer. J. Math. **78** (1956), 78–116; MR **18**, 600.] Soient I un anneau de Dedekind, $\mathfrak{o}$ une algèbre affine normale sur I, et L' un corps de fonctions sur I; l'auteur étudie la généralisation suivante du 14-ième problème de Hilbert: "(1) Est-ce que $\mathfrak{o}\cap L'$ est une algèbre affine?" Il montre que la réponse est affirmative lorsque $\dim(\mathfrak{o}\cap L')\leqq 2$, généralisant ainsi un résultat de Zariski [relatif au cas où I est un corps; cf. Zariski, Bull. Sci. Math. (2) **78** (1954), 155–168; MR **16**, 398]. Etant donnés un anneau intègre $\mathfrak{o}$ et un idéal $\mathfrak{a}$ de $\mathfrak{o}$, la réunion des idéaux fractionnaires $(\mathfrak{o}:\mathfrak{a}^n)$ est un sous-anneau $\bar{\mathfrak{o}}$ du corps des fractions de $\mathfrak{o}$, appelé le $\mathfrak{a}$-transformé de $\mathfrak{o}$; étude de cette notion. L'énoncé (1) est équivalent à chacun des suivants: "(2) Tout $\mathfrak{a}$-transformé d'une algèbre affine normale est-il une algèbre affine?" "(3) Etant donnés une algèbre affine normale $\mathfrak{o}'$ sur I, son corps des fractions L', deux éléments a_1, a_2 de $\mathfrak{o}'$ et deux éléments t_1 et t_2 transcendants sur L' et vérifiant $a_1t_1+a_2t_2=1$, est-ce que $\mathfrak{o}'[t_1, t_2]\cap L'$ est une algèbre affine?". Etant donné un modèle M sur I, on dit qu'un modèle affine M' est associé à M si $M\subset M'$ et si les localités de rang 1 de M et de M' sont les mêmes (autrement dit, si M et M' coincident en codimension 1). Soient D une partie fermée d'un modèle affine A et $\mathfrak{a}$ un idéal définissant D dans l'algèbre affine $\mathfrak{o}$ de A; pour que le $\mathfrak{a}$-transformé $\bar{\mathfrak{o}}$ de $\mathfrak{o}$ soit une algèbre affine, il faut et il suffit que $A-D$ admette un modèle affine associé (qui est alors défini par $\bar{\mathfrak{o}}$); on montre que cette condition est locale, c'est à dire qu'elle équivaut au fait que, pour toute localité $P\in D$, le $\mathfrak{a}P$-transformé de P est engendré sur P par un nombre fini d'éléments. Pour que $A-D$ soit un modèle affine, il faut et il suffit que $1\in\mathfrak{a}\bar{\mathfrak{o}}$, et D est alors divisoriel (i.e., a toutes ses composantes de codimension 1); lorsque, pour tout $P\in D$, il existe un idéal $\mathfrak{a}(P)$ de $\mathfrak{o}$ définissant D et tel que $\mathfrak{a}(P)P$ soit principal (en particulier si D est partout localement linéairement équivalent à 0), alors $A-D$ est un modèle affine. Ceci étant, l'auteur obtient le résultat annoncé au début en montrant que, si D est une partie fermée d'un modèle affine A de dimension 2 ($D\neq A$), alors D admet un modèle affine associé (de plus, si D est divisoriel, alors $A-D$ lui-même est affine): on se ramène d'abord au cas où A est normal, puis à celui où D est divisoriel; il s'agit alors de montrer que $1\in\mathfrak{a}\bar{\mathfrak{o}}$; un raisonnement de nature purement algébrique (utilisant le théorème de Krull-Akizuki, un théorème de Cohen, et le théorème d'irréductibilité analytique) montre que $1\notin\mathfrak{a}\bar{\mathfrak{o}}$ implique contradiction. Le mémoire étudie aussi l'effet de l'extension de l'anneau de base, et donne l'exemple d'un modèle affine A de dimension 3 et d'une partie fermée divisorielle D de A tels que $A-D$ ne soit pas affine (A est le cône projetant une quadrique projective, et D le plan projetant une des génératrices de celle-ci). *P. Samuel.*

Nagata, Masayoshi. On the imbedding problem of abstract varieties in projective varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 71–82.

Même terminologie que ci-dessus. Soient K un corps, x, y, z, w des indéterminées sur K, A et A' les modèles affines définis par $\mathfrak{o}=K[x, y, z, w]$ et $\mathfrak{o}'=K[x', y', z', w']$, où $x'=y/u^2$, $y'=u^2x$, $z'=w/u^2$, $w'=u^2z$ et $u=xy+zw$. Alors $M=A\cup A'$ est un modèle non singulier. Les localités $P=\mathfrak{o}_{(y,w)}$ et $P'=\mathfrak{o}'_{(y',w')}$ sont telles que $P\cap P'$ est un anneau local; donc toute fonction rationnelle f sur M qui est régulière en P et P' prend la même valeur en ces deux points, ce qui montre que M ne peut être plongé dans un modèle projectif (N.B.: le modèle M n'est pas complet, mais l'auteur a ultérieurement trouvé un exemple d'un modèle complet qui n'est pas projectif). L'auteur donne ensuite une condition pour qu'un modèle puisse être plongé dans un modèle projectif, et quelques propriétés des transformées quadratiques et monoïdales. *P. Samuel.*

Severi, Francesco. Les irrégularités des variétés algébriques, et les systèmes adjoints. C. R. Acad. Sci. Paris **244** (1957), 2333–2336.

The r-dimensional irregularity of a non-singular algebraic variety M_r of dimension r is the difference P_g-P_a between the geometric and arithmetic genera of M_r. Kodaira [Ann. of Math. (2) **59** (1954), 86–134; MR **16**, 617] proved the result, conjectured by the author [Rend. Circ. Mat. Palermo **28** (1909), 33–87], that $P_a=i_r-i_{r-1}+\cdots+(-1)^{r-1}i_1$, where i_k is the number of k-ple integrals of the first kind on M_r. Let $|A|$ be an ample linear system on M_r, and V_k a generic k-dimensional characteristic variety of $|A|$. It is known that the number of s-ple integrals of the first kind on V_k is equal to i_s for $s=1, \cdots, k-1$, $2\leqq k\leqq r-1$. Thus the irregularity of V_k is an invariant of M_r, which the author terms the k-dimensional irregularity of M_r. Since the arithmetic and geometric genera of a variety can be defined algebra-geometrically, the same is true of the irregularities of M_r. Another definition can be given as follows. A subvariety V_k of M_r is said to be topologically general if every cycle on M_r of topological dimension not

exceeding k is homologous to a cycle on V_k. In the class of topologically general V_k whose $(k-1)$-dimensional irregularity is equal to that of M_r, the k-dimensional irregularity attains a minimum value, which is the k-dimensional irregularity of M_r. Finally, using the Riemann-Roch theorem for adjoint systems [Kodaira, Contributions to the theory of Riemann surfaces, Princeton, 1953, pp. 247–264; MR **15**, 351], the author proves that if A is a hypersurface of M_r, whose $(r-1)$-dimensional irregularity is equal to that of M_r, then the difference between the virtual and effective dimensions of the system adjoint to A is equal to the number of $(r-1)$-forms of the first kind on M_r which vanish on A. *J. A. Todd.*

Schmid, Josef. Ein Beweis eines Dimensionssatzes der algebraischen Geometrie. Arch. Math. **8** (1957), 39–42.

The author gives a new demonstration of the well-known theorem that the intersection of an irreducible algebraic variety V, of dimension r, with a hypersurface H, is composed of components such that the isolated ones amongst them have all the same dimension $r-1$. His demonstration follows the methods of the American school of algebraic geometry. *M. Piazzola-Beloch.*

★**Morikawa, Hisasi. Cycles on algebraic varieties.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 245–247. Science Council of Japan, Tokyo, 1956.

This is a short note about rational homology groups on a non-singular algebraic variety in projective space. There are no proofs, and it is assumed by the reviewer that further details will appear. Let V be a closed non-singular algebraic variety of dimension n imbedded in a projective space. Let V_1 and V_2 respectively denote the first and second components of the product $V\times V$. Let W_r be the generic hyperplane section of dimension r of V. By $\{\Gamma_r^1, \cdots, \Gamma_r^{B_r}\}$ is denoted a rational homology base of $H_r(V_1; Q)$ and by $\{\Gamma_r^{1*}, \cdots, \Gamma_r^{B_r*}\}$ the dual base of $H_{2n-r}(V_2; Q)$. The author denotes by A_r the algebraic cycles in $H_{2r}(V; Q)$. Several theorems are stated about the relations between these groups and the period matrices of the differential forms on V. In particular the author describes a birational invariant as follows. Let $K_{2n-2}(V; Q)$ be the submodule of $H_{2n-2}(V; Q)$ consisting of classes Z such that $I(W_2 Z \Gamma_1^{i*}\Gamma_1^{j*})=0$ $(i, j=1, 2, \cdots, B_1)$. The dimension of

$$A_{n-1}(V; Q)/A_{n-1}(V; Q)\cap K_{2n-2}(V; Q)$$

is a birational invariant.

It is noted that if $\Omega^{(1,0)}$ is the period matrix of harmonic forms of type $(1, 0)$ over the base chosen, and if $(S_1, \cdots, S_k)$ is a base of the module of rational matrices $S=(s_{ij})$ such that $\sum s_{ij}\Gamma_1^{i*}\Gamma_1^{j*}=0$, then there is an isomorphism of $A_{n-1}(V; Q)/A_{n-1}(V; Q)\cap K_{2n-2}(V; Q)$ onto the module of rational matrices M satisfying (i) $\Omega^{(1,0)}M=\Lambda\Omega^{(1,0)}$ with a matrix Λ, (ii) $\mathrm{Sp}\, S_\nu M(I(W_1\Gamma_1^{i*}\Gamma_1^{j*}))=0$ $(\nu=1, 2, \cdots, k)$. Similar results are stated for forms of type (p, q).

P. E. Conner (Ann Arbor, Mich.).

Horrocks, G. On the relation of S-functions to Schubert varieties. Proc. London Math. Soc. (3) **7** (1957), 265–280.

The formulae for the multiplication of S-functions and the formulae for the products of Schubert-conditions are formally identical when the convention is made that meaningless condition-symbols are to be interpreted as zero. This paper goes some way towards explaining this relationship for complex Grassmannians. Extensive use is made of modern theories, of the work of de Rham, Chern, E. Cartan, Borel and Serre, and the interest of the paper lies in the skilful application of these theories to the Grassmannian, which is a symmetric homogeneous manifold under a compact group of transformations. The author modestly concludes by sighing, "the relationship between Schubert varieties and the irreducible representations of the general linear group is still very much of a mystery." *D. Pedoe* (Khartoum).

Kirby, David. Invarianti topologici d'un insieme di elementi differenziali curvilinei. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 189–193.

In Part I [same Rend. (8) **21** (1956), 66–71; MR **18**, 672] the author studied the topological invariants of a system of differential elements at a point P of an algebraic variety V_n of $n\geqq 3$ dimensions. In the present work, the case of $n=2$ dimensions is discussed. The most interesting situation appears when the differential elements all have an algebraic singularity at the point P. By the methods of algebraic geometry, the differential elements are classified into species and reduced to canonical forms. In the case of the real (x, y)-plane, this work was accomplished by Kasner and De Cicco [Trans. Amer. Math. Soc. **51** (1942), 232–254; MR **3**, 306]. *J. De Cicco.*

Kirby, David. Intorno alla classificazione dei rami e dei loro centri associati di proiezione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 179–184.

The present work is concerned with the study of the canonical forms of power series in a single indeterminate. From an algebraic viewpoint, some contributions to this subject were made by Cahit Arf [Proc. London Math. Soc. (2) **50** (1948), 256–287; MR **11**, 205]. Here some of these theorems and their extensions are treated from a geometrical standpoint. The general formulation of the problem is as follows. If K is an algebraically closed field, in particular, the complex number system, $K\{x_1, x_2, \cdots, x_N\}$ denotes the ring of formal power series in the N indeterminates x_k $(k=1, 2, \cdots, N)$ with coefficients in K. The admissable transformations T which map an element of $K\{y_1, y_2, \cdots, y_M\}$ into an element of $K\{x_1, x_2, \cdots, x_N\}$ are of the form: $x_k=f_k(y)$ $(k=1, 2, \cdots, N)$, where the $f_k(y)=f_k(y_1, y_2, \cdots, y_M)$ are elements of $K\{y_1, y_2, \cdots, y_M\}$, of positive order. The problem is to reduce a given element of $K\{x_1, x_2, \cdots, x_N\}$ to canonical form by means of such transformations T. Here the author limits himself to the case of $M=1$. For example, it is shown that a ring $H(t)$ is finitely generated if and only if it is complete, and that the order of the elements of the base of an $H(t)$ is independent of the base. *J. De Cicco* (Chicago, Ill.).

Turri, Tullio. Osservazioni a nota sulle trasformazioni piane cicliche di de Jonquières. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 142–146.

La proposition de Manara [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18**(87) (1954), 115–129; MR **17**, 192] qu'une transformation T de de Jonquières cyclique de période n subordonnant dans le faisceau invariant de droites une projectivité de même période n est birationnellement équivalent à une transformation de de Jonquières ayant un faisceau de droites unies n'est pas en général vérifiée. En fait, les transformations cycliques de de Jonquières de période >2 se réduisent à des homo-

graphies cycliques (non forcément homologiques), à l'exception de celles qui sont le produit d'une transformation I_2 de de Jonquières de période 2, à courbe hyperelliptique de points unis, par une homologie cyclique permutable avec la I_2. *B. d'Orgeval* (Dijon).

Turri, Tullio. Sulla classificazione delle trasformazioni birazionali involutorie dello spazio. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 129–141.

Classification des transformations birationnelles involutives de l'espace pour lesquelles existe une surface de points unis. Pour de telles transformations T, il existe des systèmes linéaires complets par rapport à la base de surfaces réguliéres F_n d'ordre n invariantes par T de dimension ≥ 3, d'où résulte que le système des F_n peut se transformer: a) si F_n rationnelle, dans le système des quadriques passant par six points; b) si F_n non rationnelle (donc non réglée), si $n>4$, par l'intermédiaire des adjointes successives F_{n-4}, F_{n-8}, $\cdots$, soit à un système au moins ∞^2 de surfaces rationnelles invariantes par T, soit à un système de quartiques non rationnelles de dimension 3 invariantes par T; à ce cas se raméne aussi l'hypothése $n=4$. Le produit de deux telles transformations ne possède pas en général de surfaces de points unis, mais le produit de deux transformations dotées de surfaces de points unis est permutable. Les transformations qui ne possèdent pas de surfaces de point unis sont ou des produits du type précédent, ou d'un type dont on ne connaît qu'un exemple, celui de la transformation (3, 3) construite au moyen des polarités par rapport à un réseau de quadriques. *B. d'Orgeval* (Dijon).

Trempont, Jacques. Sur une transformation birationnelle du plan et sa représentation hyperspatiale. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 442–451.

Le système des C_{4n+1} passant $2n$ fois en quatre points A_i et deux fois en $2n$ points B_k avec tangentes variables constitue un réseau homaloïdal permettant de définir une transformation de Cremona, dont l'inverse se définit par un système analogue. Les courbes fondamentales sont formées de quatre courbes d'ordre $2n$ $(3A^n, A_i^{n-1}, 2nB)$ et des $2n$ coniques $(4A, B_k)$. La considération des $H_{2n+1}(4A^n, 2nB)$ auxquelles correspondent des courbes analogues permet d'associer à la transformation une surface F de S^{n+2}, d'ordre $2n+1$ à sections de genre n, images des couples homologues du plan; sur cette surface aux courbes fondamentales du plan correspondent deux groupes de quatre courbes rationnelles d'ordre n et de $2n$ droites, dont la configuration est étudiée. Cette surface appartient à $\frac{1}{2}n(n-1)$ quadriques indépendantes qui contiennent aussi la V_n^3 lieu des plans des coniques appartenant à F. Si la transformation T est involutive, la F est conservée par une homographie harmonique qui y engendre une involution. De l'étude de cette homographie on conclut que la courbe unie de T est une C_{2n-1} passant $n-1$ fois aux points A; il y a également trois points unis isolés qui sont les points diagonaux du quadrangle des A. Les coniques du faisceau défini par les A touchent la courbe unie aux points B. Cette involution peut se représenter par une surface rationnelle normale de S^n, dont la courbe de diramation est formée de trois génératrices et d'une C_{2n-1} unisécante des génératrices. *B. d'Orgeval.*

See also: Differential Geometry: Segre; Thom.

NUMERICAL ANALYSIS

★**Kunz, Kaiser S. Numerical analysis.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. xv+381 pp. $8.00.

Contents are as follows: 1. The real roots of an equation. 2. Roots of polynomial equations. 3. Finite-difference tables and the theory of interpolation. 4. Central-difference interpolation formulas. 5. Lagrange's interpolation formula and inverse interpolation. 6. Summation of series. 7. Numerical differentiation and numerical integration. 8. Numerical solution of ordinary differential equations: methods of starting the solution. 9. Solution of ordinary differential equations: methods for continuing the solution. 10. Simultaneous equations and determinants. 11. Interpolation in tables of two or more variables. 12. Expression of a partial differential equation as a partial difference equation. 13. Solution of partial differential equations of the elliptic type. 14. Solution of partial differential equations of the parabolic and hyperbolic types. 15. Integral equations. Appendix A. Estimation of error in numerical computation. Appendix B. References.

The treatment is elementary and graphic, with numerous illustrative examples. Although it is written primarily for the engineer, nevertheless, apart from an occasional "it can be shown", there is usually an effort to justify the methods. Unusual features are chapters 6 and 11 in particular, and extensive use of the lozenge diagram, with application to differentiation and quadrature. Curiously, neither in 6 nor elsewhere is the powerful δ^2-process mentioned. This is perhaps because iterative methods receive only cursory attention, although Shanks and others have drawn attention to the utility of the δ^2-process for summing series.

Appendix B is very short, only 8 items, but additional references occur in footnotes, and there is a real effort to attribute techniques to their inventors.

Proofreading seems to have been careful on the whole, but on p. 21 there is an omission of a necessary subscript on an entire line, and on p. 359 there is an annoying oscillation between "e" and "ε" for designating errors. And it seems a pity that authors are unwilling to accept the useful term "blunder" introduced by von Neumann and Goldstine to designate malfunctioning, reserving the term "error", in computational processes, for those deviations alone that arise from failure to reach the limit. Finally, the term "expression" in 12 could be interpreted as meaning that it can be done without error.

A. S. Householder (Oak Ridge, Tenn.).

Numerical Methods

★**Shaw, F. S. An introduction to relaxation methods.** Dover Publications, Inc., New York, N. Y., 1958. 396 pp. $2.45.

A paperbound reprint of the book reviewed in MR **15**, 353.

Schröder, Johann. Nichtlineare Majoranten beim Verfahren der schrittweisen Näherung. Arch. Math. **7** (1957), 471–484.

Quite general sufficient conditions for the convergence of the iteration scheme (1) $u_{n+1}=Tu_n$ to the unique solution of $u=Tu$ are given. The operator T is defined in a space $\mathfrak{R}$ with elements u, v, $\cdots$. For every pair of ele-

ments u, v in $\mathfrak{R}$ a "distance" $\rho(u, v)$ is defined. The objects $\rho(u, v)$ are elements in a linear, partially ordered space $\mathfrak{R}$. The basis for estimating the convergence of (1) is the inequality ("Lipschitz condition") (2) $\rho(Tu, Tv) \leqq Q\rho(u, v)$ where the operator Q is defined in $\mathfrak{R}$ as a function of the elements $[\rho(u, w), \rho(v, w)]$ for some fixed element w of $\mathfrak{R}$. Furthermore Q is assumed to be positive and monotonically increasing, i.e. (3) $Q(\rho_1, \rho_2)\rho_3 \leqq Q(\rho_1^*, \rho_2^*)\rho_3^*$ if $\rho_1 \leqq \rho_1^*$, $\rho_2 \leqq \rho_2^*$ and $\rho_3 \leqq \rho_3^*$. Finally Q is assumed to be the difference quotient of some continuous operator P defined in $\mathfrak{R}$ in the form (4) $Q(\rho_1, \rho_2)(\rho_1 - \rho_2) = P\rho_1 - P\rho_2$. Under these definitions the nonlinear operator P is called a "majorant" of the operator T. Sufficient conditions for the convergence of (1) are expressed in terms of the convergence of a related sequence involving the operator P. This work generalizes and makes more readily applicable to numerical analysis the earlier work of Kantorovitch [Acta Math. **71** (1939), 63–97; MR **1**, 18] as well as earlier work of the present author [Z. Angew. Math. Mech. **36** (1956), 168–181; Math. Z. **66** (1956), 111–116; MR **18**, 152, 765]. *E. Isaacson* (New York, N.Y.).

★ **Ottaviani, Giuseppe. Sulla risoluzione di una equazione con il metodo di iterazione.** Scritti matematici in onore di Filippo Sibirani, pp. 195–199. Cesare Zuffi, Bologna, 1957.

Proof of the following theorem. Let $f(x)$ be real and continuous in the closed interval $\langle a, b\rangle$ and $a \leqq f(x) \leqq b$. The iteration sequence x_0, $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\cdots$ has one and the same limit α in $\langle a, b\rangle$ for whatever x_0 in $\langle a, b\rangle$ if and only if for all equations

(*) $\qquad f(x) = x,\ f(f(x)) = x,\ f(f(f(x))) = x,\ \cdots$

there is one and only one solution $x = \alpha$ in $\langle a, b\rangle$. The necessity of the condition follows easily in the indirect way, assuming another solution, different from α. The sufficiency proof is based on the lemma stating that in the case of the uniqueness of the solution of all equations (*) the iteration sequence is monotonic. {Reviewer's remark: A more practical condition from which this theorem follows immediately may be found in W. A. Coppel, Proc. Cambridge Philos. Soc. **51** (1955), 41–43; MR **16**, 577.}
H. Schwerdtfeger (Montreal).

Bauer, Friedrich L. Das Verfahren der Treppeniteration und verwandte Verfahren zur Lösung algebraischer Eigenwertprobleme. Z. Angew. Math. Phys. **8** (1957), 214–235.

This paper discusses two methods which are similar to the *LR* Transformation of Rutishauser and are called "stabilized iteration" and "mutually stabilized bi-iteration". While both methods seem to have much to recommend them as practical schemes, the author suggests "A certain mixed process, based on Rutishauser and the author's ideas, seems to be the optimum way with regard to both security and numerical labor".

The paper contains proofs of the convergence and other properties of the two methods. *R. Hamming.*

Kitagawa, Tosio. Some aspects of stochastically approximative analysis. Bull. Math. Statist. **6** (1956), 109–129.

In his introduction the author discusses in very general terms the choice, from among the various standard procedures of numerical analysis, of the treatment to be applied to random data of various kinds. The rest of the paper is devoted to illustrations. In Part I, the data are a realization $\{y_i\}$ of a sequence $\{Y_i\}$, $(i = 1, 2, \cdots, n)$ of random variables, with means $m_i = m_0 + A_i$ and variances B_i^2, respectively, and a linear function $\sum_1^n c_i y_i$ of the y_i's is to be chosen as an approximation to m_0. The "risk" is measured by the mean square error

$$R(A, B; c) = \Big(\sum_1^n c_i A_i\Big)^2 + \sum_1^n c_i^2 B_i^2.$$

The A_i's and B_i's are unknown, but the point $(A, B) = (A_1, \cdots, A_n, B_1, \cdots, B_n)$ is confined to a region Ω. The author's procedure is suggested by the theory of games: he seeks a minimax solution, that is, a point $c = (c_1, \cdots, c_n)$ in the hyperplane C: $\sum_1^n c_i = 1$ such that $\max_\Omega \min_C R(A, B; c)$ is equal to $\min_C \max_\Omega R(A, B; c)$. He shows (Theorem 1.1) that if Ω is the region

$$\Omega_1\colon \sum_1^n A_i^2/B_i^2 = \Theta_0,\ B_i^2 \leqq B_0^2/\phi_i\ (i = 1, \cdots, n),$$

where $\Theta_0, B_0, \phi_1, \cdots, \phi_n$ are given constants, then the minimax solution is $c_i = \phi_i/\sum_1^n \phi_i$. In Theorem 1.2 he generalizes this to a case where the A_i's and B_i's are themselves random variables. In Parts II and III, the author contents himself with preliminary analyses of various problems and in most cases simply obtains formulae for the mean square errors concerned. He treats in this way some problems of interpolation and curve-fitting (including cases where formulae of increasing complexity are tried successively) under various plausible assumptions about the random character of the function to be fitted.
H. P. Mulholland (Exeter).

Zajta, A. Über die Iteration der Potenzreihen. Z. Angew. Math. Mech. **37** (1957), 153–155.

Let $P(x) = \sum_{n=0}^\infty a_n x^{n+1}$, and let $P^m(x) = \sum_{n=0}^\infty A_n x^{n+1}$ be the series obtained by m-fold iteration (i.e. composition) of P. The author presents a method for calculating the coefficients of $P^m(x)$. The method is based on a familiar group-theoretic property, namely, if P and T are conjugate elements in a group such that $P = fTf^{-1}$, then $P^m = fT^m f^{-1}$. To apply this property to the problem at hand, let $T(x)$ be the similarity transformation, εx, $\varepsilon \neq 0$. Let $f(x) = x + \sum_1^\infty c_n x^{n+1}$ and let $f^{-1}(x)$ be the reversion of $f(x)$. If $f(x)$ is such that (1) $P(x) = f(\varepsilon f^{-1}(x))$, then (2) $P^m(x) = f(\varepsilon^m f^{-1}(x))$. Using formulas for the reversion and power of a power series, the author proceeds formally to derive expressions for the coefficients of $f(\varepsilon f^{-1}(x))$. These expressions are used in (1) to determine the c_n from the a_n, and again in (2) to determine the A_n from the c_n. The method fails if a_0 is zero or a root of unity. The author's formula for the reversion of the power series, $f(x)$, is presented here without proof as a result obtained in a paper of the author to appear soon. It is as follows: $f^{-1}(x) = \sum_0^\infty b_n x^{n+1}$, where

$$b_n = \sum_P [(-1)^P (n+P)! \cdot \prod_{k=1}^n (c_k^{i_k}/i_k!)/(n+1)!],$$

and the summation extends over all $P = \sum_{k=1}^n i_k$, where the i_k are non-negative integral solutions of (3) $i_1 + 2i_2 + \cdots + ni_n = n$. The reviewer would like to point out that this formula for b_n follows immediately from the Lagrange reversion formula and a formula for the nth derivative of a composite function, due to Faà de Bruno [Quart. J. Pure Appl. Math. **1** (1857), 359–360]. The formula of de Bruno, with $u = \phi(\psi(x))$, is $d^n u/dx^n = \sum n!(d^P\phi/d\psi^P)\prod_{k=1}^n D_k$, where $D_k = (d^k\psi/dx^k)^{i_k}/i_k!k!$, the summation being extended over the solutions of (3) as before. *E. K. Blum* (Los Angeles, Calif.).

Červená, Alena. Eine Bemerkung über die Lösungsfrage eines speziellen Systems von Ungleichungen durch positive Zahlen. Časopis Pěst. Mat. 82 (1957), 335–341. (Czech. Russian and German summaries)

Eigentliches Ziel der vorgelegten Arbeit ist der Beweis des folgenden Satzes: Notwendige und hinreichende Bedingung dazu, dass ein System von Ungleichungen

$$\begin{aligned}
&\alpha_1 C_{11}-\alpha_2 C_{12}-\alpha_3 C_{13}-\cdots-\alpha_n C_{1n}>0,\\
&-\alpha_1 C_{21}+\alpha_2 C_{22}-\alpha_3 C_{23}-\cdots-\alpha_n C_{2n}>0,\\
&\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\\
&-\alpha_1 C_{n1}-\alpha_2 C_{n2}-\alpha_3 C_{n3}-\cdots+\alpha_n C_{nn}>0,
\end{aligned}$$

wo alle Konstanten C_{ik} positiv sind, eine Lösung $\alpha_1, \alpha_2, \cdots, \alpha_n$ besitzt, wo $\alpha_i>0$ für $i=1, 2, \cdots, n$, ist die folgende: a) Das Produkt $C_{11}C_{22}\cdots C_{nn}$ besitzt den grössten Wert zwischen allen Produkten von dem Typ $C_{1i_1}C_{2i_2}\cdots C_{ni_n}$, wo $i_1, i_2, \cdots, i_n$ eine Permutation der Zahlen 1, 2, $\cdots$, n ist. b) Die Determinante des genannten Systems von Ungleichungen ist grösser als Null.

In Bemerkungen zum zitierten Satz zeigt man, dass man durch die Vernachlässigung der Voraussetzung darüber, dass alle Konstanten C_{ik} positiv sind, zu einem nicht mehr richtigen Satz gelangt. Weiter wird die gegenseitige Abhängigheit der Bedingungen a) und b) untersucht. Für $n=2$ ist die Bedingung a) Folge von b) und umgekehrt; für $n=3$ ist a) Folge von b), aber nicht mehr b) Folge von a), für $n\geqq 4$ sind die beiden Bedingungen gegenseitig unabhängig. *Zusammenfassung des Autors.*

Kimura, Hitosi. An approximation method in numerical computation of the Leontief's open input-output model. Ann. Inst. Statist. Math., Tokyo 7 (1956), 115–122.

Kimura, Hitosi. On the errors of outputs due to errors of technical coefficients in Leontief's open input-output models. Ann. Inst. Statist. Math., Tokyo 7 (1956), 205–213.

In Leontief's input-output model, the level of outputs $X_i=\sum a_{ij}X_j+Y_i$ may be approximated by $X_i^{(1)}=(1-a_{ii})^{-1}(Y_i+\sum_{i\neq j} a_{ij}X_j')$,

$$X_i^{(2)}=X_i^{(1)}\left(1+\frac{X_i'-(1-a_{jj})^{-1}Y_i'}{X_i}\cdot\frac{\sum(1-a_{jj})^{-1}(Y_j-Y_j')}{\sum X_j'}\right),$$

$$X_i^{(3)}=X_i^{(1)}+X_i'\cdot\frac{X_i'-(1-a_{ii})^{-1}X_i'}{X_i'}\cdot\frac{\sum(1-a_{jj})^{-1}(Y_j-Y_j')}{\sum X_j'},$$

where X_i's and Y_i's are the levels of outputs and final demands used to calculate a_{ij}. Estimates are also given of errors in approximation. *S. Ichimura* (Osaka).

Bertram, G. Fehlerabschätzung für das Ritz-Galerkinsche Verfahren bei Eigenwertproblemen. Z. Angew. Math. Mech. 37 (1957), 191–201. (English, French and Russian summaries)

A typical result is: let ϕ_ρ be the eigenfunction corresponding to the eigenvalue ω_ρ of the self-adjoint differential equation $M[y]=\lambda y$ of order $2m$ with boundary conditions $U_\mu[y]=0$ ($\mu=1, \cdots, 2m$). If the first r eigenfunctions ϕ_ρ are used as coordinate functions in applying the Rayleigh-Ritz method to $M[y]=\lambda q(x)y$ with the same boundary conditions, then

$$\lambda_{\nu,s}-\lambda_\nu\leqq\lambda_{\nu,r}\frac{\lambda_{\nu,r}\,q_{\max}\left\{\int_a^b K(x,x)dx-\sum_{\varrho=1}^{r}\omega_\rho^{-1}\right\}}{1+\lambda_{\nu,r}\cdot q_{\max}\left\{\int_a^b K(x,x)dx-\sum_{\varrho=1}^{r}\omega_\rho^{-1}\right\}},$$

where $\lambda_{\nu,r}$ is the rth approximation to the eigenvalue λ_ν and $K(x,\xi)$ is Green's function for the first problem $K(x,\xi)=\sum_{\varrho=1}^{\infty}\phi_\rho(x)\phi_\rho(\xi)\omega_\rho^{-1}$. Convergence and asymptotic relations can be investigated from this formula.

R. C. T. Smith (Armidale).

Grohne, Diether. Bemerkung zur Erweiterung des Verfahrens von Newton-Raphson auf die Berechnung einer mehrfachen Nullstelle. Z. Angew. Math. Mech. 37 (1957), 233.

Ghizzetti, Aldo. Sulle formule di quadratura. Rend. Sem. Mat. Fis. Milano 26 (1954–55), 45–60 (1957).

A quite general quadrature formula for the integration of the product of two functions is given in the form

$$\int_a^b fg\,dx=\sum_{h=0}^{n-1}\sum_{i=1}^{m}A_{hi}f^{(h)}(x_i)+R$$

in which the m x's subdivide the interval (a, b). The coefficients A_{hi} depend on g via a more or less arbitrary $(n-1)$-st order differential operator E and its adjoint. Likewise the remainder R depends on both f and g and these two operators. Green's formula is used in the derivation. Much of the paper is devoted to the deduction of previously known special cases from the general result.

D. H. Lehmer (Berkeley, Calif.).

Quade, W. Numerische Integration von gewöhnlichen Differentialgleichungen durch Interpolation nach Hermite. Z. Angew. Math. Mech. 37 (1957), 161–169. (English, French and Russian summaries)

The author essentially exploits the fact that, if $y(x)=\sum_{k=0}^{2n+1}a_k(y_0, \cdots, y_n; y_0', \cdots, y_n')x^k$ denotes the polynomial of degree $2n+1$ or less for which $y(x_i)=y_i$ and $y'(x_i)=y_i'$, then a_r is a linear function of its arguments which vanishes identically if $y(x)$ is a polynomial of degree inferior to r. In particular, the condition $a_{2n+1}=0$ constitutes a formula representing y_n as a linear combination of $y_{n-1}, \cdots, y_0$ and $y_n', \cdots, y_0'$ with a degree of precision of at least $2n$. Several such formulas, and related ones, are presented together with leading terms in expansions of the relevant errors. *F. B. Hildebrand.*

Albrecht, Julius. Zum Differenzenverfahren bei parabolischen Differentialgleichungen. Z. Angew. Math. Mech. 37 (1957), 202–212. (English, French and Russian summaries)

Various finite-difference simulations of the one- and two-dimensional heat flow equations and of the bar-vibration equation are derived and tested for stability by conventional methods. Triangular and hexagonal nets are included. *F. B. Hildebrand* (Cambridge, Mass.).

Manfredi, Bianca. Soluzioni numeriche in problemi pluridimensionali lineari di conduzione del calore. Riv. Mat. Univ. Parma 7 (1956), 51–77.

The author treats problems of convergence and stability relevant to finite-difference simulation of the equation $\partial U/\partial t=\sum_1^r \partial^2 U/\partial x_i^2$, for the most part generalizing known results for the case $r=1$.

F. B. Hildebrand (Cambridge, Mass.).

★ **Witting, Hermann. Uber die numerische Lösung parabolischer Differentialgleichungen.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 127–132. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

The author investigates relations between the grid

spacings h and l (defined by $x_r=x_0+rh$, $y_s=a+sl$) that will insure favorable (i.e., damped) error propagation for finite difference equations approximating parabolic partial differential equations. In case the original differential equation is not stable, damped error propagation becomes impossible, and the best that one can ask is that error propagation be no worse for the finite difference equation than for the original differential equation. For a large class of differential equations this requirement leads to relations (independent of boundary conditions) between the grid spacings h and l. The method of investigation depends on the Fourier expansion of arbitrary solutions in terms of a fundamental set of solutions, so that only the growth of the fundamental solutions (of very special form) need be considered. *R. B. Davis* (Syracuse, N.Y.).

Durand, Emile. L'approximation du quatrième ordre dans le calcul numérique des systèmes de révolution. C. R. Acad. Sci. Paris **244** (1957), 2355–2358.

A finite difference approximation for the partial differential equation

$$\left[\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{k}{r}\frac{\partial}{\partial r}\right]\psi(r,z)=f(r,z)$$

is developed which is exact up to and including fourth order terms. The expressions are obtained by the well known technique of using Taylor series. They are limited in their practical applicability since they involve the partial derivatives $\partial f/\partial r$, $\partial^2 f/\partial r^2$, $\partial^2 f/\partial z^2$ which may not always exist. *U. W. Hochstrasser* (Lawrence, Kans.).

Freeman, John C., Jr.; and Baer, Ledolph. Pseudo-characteristics. Trans. Amer. Geophys. Union **38** (1957), 65–67.

Pour un système d'équations aux dérivées partielles hyperboliques à deux variables l'auteur appelle pseudo-caractéristiques des courbes sur lesquelles les inconnues satisfont aux relations fonctionnelles qu'elles vérifient sur les caractéristiques du système homogène associé. Il indique, dans le cas d'un écoulement à une dimension d'eau ventée dans un bassin peu profond, comment tracer ces lignes $u+2\sqrt{(gh)}=c^{te}$ et $u-2\sqrt{(gh)}=c^{te}$, u et h vitesse et hauteur du fluide, g accélération de la pesanteur). Exemple numérique et diagramme.

Y. Fourès-Bruhat (Marseille).

Boxer, Rubin. A note on numerical transform calculus. Proc. I. R. E. **45** (1957), 1401–1406.

Many of the numerical transform methods, developed for the solution of integrodifferential equations, exhibit difficulties when discontinuities occur at the origin. In this paper a procedure for including initial conditions is derived which effectively takes origin discontinuities into account. Although shown to be applicable to a number of transform methods, the derivation emphasizes application to the z-form integration operators because of their relatively greater accuracy. Numerical examples illustrating the solution of a constant coefficient and a time-varying problem are given. The technique is equally applicable to the solution of nonlinear problems.

Author's summary.

Moon, Parry; and Spencer, Domina Eberle. Errors in the solution of integral equations. J. Franklin Inst. **264** (1957), 29–41.

The authors consider the Fredholm integral equation $y(s)=\lambda\int_a^b K(s,t)y(t)dt+f(s)$ which they propose to solve by means of approximations to $K(s,t)$ and $f(s)$. They set $f=f_0+h$, $y=y_0+\delta$, $K=K_0+\Delta$ and solve

$$y_0(s)=\lambda\int_a^b K_0(s,t)y_0(t)dt+f_0(s).$$

The correction δ is estimated from the equation $\delta(s)=\lambda\int K(s,t)\cdot\delta(t)dt+g(s)$, where $g(s)=h(s)+\lambda\int\Delta(s,t)f_0(t)dt$. The method is applied to the problem of multiple reflection of radiation in circular cylinders. This leads to an integral equation with the kernel

$$K(s,t)=\tfrac{1}{2}\left[1-|s-t|\,\frac{(s-t)^2+6}{\{(s-t)^2+4\}^{3/2}}\right]$$

which is approximated by $K_0=A\exp(-\alpha|s-t|)$ with suitable constants A, α. Some numerical examples are presented.

The authors claim that the method is novel and promising, especially if one approximates a kernel by a one term exponential function. They also feel that the problem of estimating the magnitude of the error $-\delta$ has not yet been treated adequately in the literature. In order to estimate the magnitude they propose three procedures, one of them being set up in the form

$$\delta_m\leqq g_m+\lambda\delta_m[\int K(s,t)dt]_m,$$

where the subcript m denotes the maximum. From this relation the bound $\delta_m\leqq g_m(1-\lambda[\int K(s,t)dt]_m)^{-1}$ is derived.

{The reviewer believes that this bound is wrong in general. It will hold only in special cases, where δ and K do not change the sign. Even in the case of positive quantities λ, δ, K it can only be true for $1-\lambda[\int K(s,t)dt]_m>0$. The other two procedures show the same lack of rigour.}

It is to be mentioned that the general method of approximation is by no means new, as can be seen from exercise No. 100, pp. 501–503 of the book by F. B. Hildebrand, "Methods of applied mathematics" [Prentice-Hall, New York, 1952; MR **15**, 204]. The approximation of a kernel by means of a one term exponential $A\exp(-\alpha|s-t|)$ will not apply unless the kernel is at least approximately of the difference-type. It seems that the significance of the paper is restricted to special numerical calculations for a special problem.

H. F. Bückner (Schenectady, N.Y.).

★ **Abramow, A. A. Über Rundungsfehler bei der Lösung linearer Gleichungssysteme.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 151–153. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. (Russian. German summary)

Der Verfasser schätzt den Einfluß von Rundungsfehlern bei der Lösung eines linearen Gleichungssystems nach dem klassischen Gaussschen Algorithmus ab. Unter der Annahme, daß die Rundungsfehler unabhängige Zufallsveränderliche mit dem Mittelwert Neun und bekannter Streuung sind, wird die Streuung der Restterme angegeben, die sich beim Einsetzen der Näherungslösungen in das System ergeben. Dabei wird die Anzahl der tatsächlich durchzuführenden Rechenoperationen berücksichtigt. Aus dem Ergebnis werden Schlüsse für die Praxis der Rechnung gezogen. *Zusammenfassung des Autors.*

★ **Rybner, Jørgen. Nomogrammer over komplekse hyperbolske funktioner. [Nomograms of complex hyperbolic functions.]** 2nd ed. Jul. Gjellerups Forlag, Copenhagen, 1955. 39 pp.+60 charts. Dan. kr. 44.00.

This work is concerned with the presentation in nomographic form of many of the functions met in electric transmission line and network theory. Included in the book are

sections on the use of the nomograms, formulas for circular and hyperbolic functions, tables of multiples of $\pi/2$, formulas for four terminal networks and transmission lines, and sixty nomograms. The material is presented in both Danish and English and provides a good ready-reference for the engineers as well as a means for making quick calculation on networks or transmission lines.

R. T. Herbst (Winston-Salem, N.C.).

Weiss, L. L. A nomogram for log-normal frequency analysis. Trans. Amer. Geophys. Union 38 (1957), 33–37.

A graphical means for determining the log normal return period X_T as a function of the mean $\bar{X}$ and the coefficient of variation C_V of the annual maximum series values from the relations,

$$\log \bar{X} = X_L + 1.15129 S_{X_L}^2,$$
$$\log C_V^2 + 1 = 2.3026 S_{X_L}^2,$$
$$\log X_T = S_{X_L} + \bar{X}_L,$$

is given.

R. T. Herbst (Winston-Salem, N.C.).

Harris, Frank E. Tables of the exponential integral Ei(x). Math. Tables Aids Comput. 11 (1957), 9–16.

Values of $-\mathrm{Ei}(-x)$ and $-e^x\mathrm{Ei}(-x)$ are tabulated to 18 significant figures and 19 decimal places respectively for $\pm x = 1(1)4(.4)8(1)50$.

The use of the recurrence formula

$$e^{-x}\mathrm{Ei}(x) = e^{-h}[e^{-(x-h)}\mathrm{Ei}(x-h)] + x^{-1}\sum_{k=0}^{\infty} x^{-k}R_k(k)$$

is recommended for interpolation and its application is further facilitated by the tabulation of the coefficients $R_n(x)$ and $e^{\pm x}$ for $x = 1, .01(.01).05, .1(.1).5$.

This article belongs to the class of tables of special functions with emphasis on interpolation and the use of recurrence relations, high accuracy being desirable in molecular structure calculations.

I. A. Stegun.

★ **Cerrillo, M. V.; and Kautz, W. H. Properties and tables of the extended Airy-Hardy integrals.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. 144 (1951). iii+61 pp.

Tables and graphs are given of the real and imaginary parts of the functions $Ah_{1,3}(B) \equiv 3^{-1/3}\mathrm{Ai}(-3^{-1/3}B)$ and $e^{\mp\pi i/3}Ah_{1,3}(Be^{\pm 2\pi i/3})$, and of the modulus and phase of $Ah_{1,3}(B)$, where Ai is the Airy integral. The ranges are arg $B = 0°(7°.5)180°$, $|B| = 0.0(0.2)4.0$, and the accuracy 7 decimals. No interpolation facilities are provided; interpolation is the arg B direction is in fact difficult. Sample checking has revealed no errors, apart from possible end-figure ones. In an introduction and an appendix, ascending series, asymptotic expansions and contour integrals are given for the functions .{In the reviewer's copy 8 pages of tables are missing, duplicates of other pages being bound in their stead.}

F. W. J. Olver.

★ **Milne-Thomson, L. M. Jacobian elliptic function tables.** Dover Publications, Inc., New York, N. Y., 1950. xi+123 pp. $1.35.

A paperbound reprint of the book reviewed in MR 13, 987.

★ **Faucher, Clovis. Tables trigonométriques contenant les valeurs naturelles des sinus et des cosinus de centigrade en centigrade du quadrant avec dix décimales.** Gauthier-Villars, Paris, 1957. 51 pp. 500 francs.

This is a surprising table to be published at this time: it has been computed by hand, using multiplication tables of Crelle and Peters and has been reproduced from manuscript. It gives $\sin x$ and $\cos x$ to $10D$ for $x = 0(10^c)50^c$, together with first differences; one hundred values of each function appear on a page. A dot attached to the last figure indicates that it should be rounded up. The tables were prepared by quadratic interpolation in the $15D$ tables of H. Andoyer [Nouvelles tables trigonométriques fundamentales, t. I, Hermann; Paris, 1915], the calculations being performed to twelve decimals. Quadratic interpolation is necessary to obtain full accuracy in the present table, and is illustrated by an example.

John Todd.

Herschbach, Dudley R. Tables of Mathieu integrals for the internal rotation problem. J. Chem. Phys. 27 (1957), 975.

See also: Geometry of Numbers: Turán. **Approximations, Orthogonal Functions:** Berman. **Partial Differential Equations:** Legras. **Topological Vector Spaces:** Fan. **Convex Domains, Integral Geometry:** Goldman and Tucker. **Numerical Analysis:** Kunz. **Computing Machines:** Ip. **Economics, Management Science:** Supnick. **Programming, Resource Allocation, Games:** Pearson.

Computing Machines

Frankel, Stanley P. The logical design of a simple general purpose computer. I.R.E. Trans. EC-6 (1957), 5–14.

Techniques used by the logical equation school of computer designers are explained by developing the logical design of the LGP-30, medium-speed computer. It uses a magnetic drum to store information and to supply certain timing signals which control the sequencing of operations. Fifteen toggles (or flipflops) determine the internal state of the computer. Each toggle is a bistable device controlled by two input lines, one for turning it on and one for turning it off. The signals on these lines are supplied from a logical net consisting of diode circuitry whose inputs are signals from the magnetic drum and the toggles themselves. Logical functions represented by this diode net thus control the sequence of states of the toggles and hence determine the behavior of the computer. Logical design consists of the derivation of the logical functions of the diode net and is the subject of this paper.

A list of sixteen instruction types, such as add, subtract, record on the drum, etc., is provided as a starting point for the logical design. Each instruction type is broken down into elementary operations which are described in terms of transitions of the toggles. These transitions are realized by including corresponding terms in the logical functions which will produce them. The entire set of logical functions is thus formed, term by term, during a detailed analysis of the instructions.

D. E. Muller (Urbana, Ill.).

★ **Grabbe, Eugene M. (editor). Automation in business and industry.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957. xix+611 pp. $10.00.

This book is an amplified version of an engineering extension course offered at the University of California in the spring of 1955. The introduction and nineteen chapters are all written by different authors so that there are, inevitably, some repetitions and inconsistencies of notation, but the range of subject matter is so broad that the overlap is not serious.

The introduction, first chapter and last two chapters

are general, tracing the history of automation, the reasons for the present day drive towards it, some of its probable effects on our economy, and methods by which it might be realized more quickly. There are seven chapters describing the basic components of automatic systems, on terminology, the fundamentals of digital and analog computers, on feedback systems and instrumentation for control, input-output equipment and analog-to-digital conversion systems. Two chapters describe the use of analog computers and numerically controlled milling machines. The other chapters describe applications of automation in particular areas; to data processing, the control of flight, the production of electronic equipment, the process control (oil and chemical) industries and industrial manufacturing.

The information in all of the chapters is detailed, voluminous, well-presented on the whole, and up-to-date. Much of it has hitherto been obtainable only by consulting a large number of technical journals, so different in scope as to hardly fall within the range of interest of one man. Bringing it all together is certainly a valuable contribution. Apparently the course for which this book forms the "Notes" arose from a "Handbook of Automation, Computation and Control" which the editor and some of authors are preparing. Until this Handbook appears this present book will serve well, and even then, it should be a useful version in abbreviated form. *C. C. Gotlieb.*

Pawlak, Z.; and Wakulicz, A. Use of expansions with a negative basis in the arithmometer of a digital computer. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 233–236, XIX–XX. (Russian summary)

The authors suggest the use of a negative basis for representing numbers in a computer in order to obtain a uniform treatment of all the bits within the arithmetic unit. The possibility of such a representation is based on the theorems that every real number α possesses an expansion with an integer basis $g<-1$ and that this expansion is unique if it is finite or in the case of infinite expansions, if the number α is not of the form:

$$\alpha=E(g^k/(1-g))+Cg^{k+1} \quad (E=\pm 1,\ C,\ k \text{ integers}).$$

The numbers satisfying this relation have two distinct infinite expansions. The algorithms for addition, subtraction, multiplication and division are discussed. These operations are somewhat more complicated than in the case of a positive basis particularly the division. Further the 0 in the case of fixed point numbers is not at the center of the interval thus avoiding the difficulty of "+0" and "−0".

The authors claim that since their system allows to treat all the bits of a number uniformly (no special sign position) the number of circuits in the arithmetic unit with different functions can be reduced.

U. W. Hochstrasser (Lawrence, Kansas).

Lloyd, S. P. Binary block coding. Bell System Tech. J. **36** (1957), 517–535.

This paper examines the problem of finding close packed error-correcting codes, and derives an additional necessary condition beyond the usual Hamming condition. The paper includes a discussion of all the cases of such codes for letter length n up to $n=150$. Perhaps the most important contribution of this paper is the successful introduction of the methods and tools of analysis into a problem in combinatorial analysis and group theory. This is accomplished via generating functions and a differential equation for a generating function. *R. W. Hamming.*

Kovásznay, Leslie S. G. An electronic analogue for supersonic flow. J. Fluid Mech. **2** (1957), 383–396.

The paper describes preliminary experiments (with indication of extensions) with an ingenious apparatus for depicting as a schlieren image the steady-state conditions of non-linear two-dimensional supersonic flow round airfoils, etc. The picture is displayed on a cathode ray tube on which the scan follows the characteristics of the quasi-linear equation of steady state supersonic flow. The correct scan is achieved by using a mask on a second cathode ray tube in order to introduce the appropriate boundary conditions. The (simplified solution) example photographs clearly show the formation of systems of shock-waves. *J. G. L. Michel* (Teddington).

Ip, S. K. An electrolytic tank as an analogue computing machine for factorizing high degree polynomials. Quart. J. Mech. Appl. Math. **10** (1957), 369–384.

In this paper the author describes the use of an electrolytic tank to determine the roots of a polynomial

$$(1) \qquad f(z)\equiv a_n z^n+a_{n-1}z^{n-1}\cdots a_1 z+a_0=0$$

The principle involved has already been described by Lucas [C. R. Acad. Sci. Paris **106** (1888), 587–589] and subsequent authors [e.g. A. R. Boothroyd, E. C. Cherry, and R. Makar, Proc. Inst. Elec. Engrs. Part I. **96** (1949), 163–177].

If $f(z)$ is of degree n, and is divided by a polynomial

$$F(z)=\prod_{s=1}^{n+1}(z-x_s)$$

of degree $n+1$, the quotient $G(z)=f(z)/F(z)$ is represented in the form $\sum_{s=1}^{n+1} k_s/(z-x_s)$. The potential function $V(z)=\sum_{s=1}^{n+1} k_s \log(z-x_s)$ is then readily set up in an electrolytic tank and the zeros of (1) may be found with a single probe as stationary points of $V(z)$. As a matter of convenience the author conformally transforms the half plane Im $z\geqq 0$ into a unit circle by writing $\zeta=-1/(z+i)$; the potential distribution is then investigated in a circular electrolytic tank.

The amount of ancillary calculation and the limited accuracy attainable (errors of a few per cent) make it unlikely that this method would be of much practical value in these days of high speed digital computers, but the process has didactic value in the field of complex variable theory. *J. G. L. Michel* (Teddington).

Lazaryan, V. A. Electric modelling of transitional regimes of the motion of rods with elastic imperfections. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 311–327. (Ukrainian. Russian summary)

Possibilities are investigated of electrically simulating transient motion regions of elastic-viscous rod and elastic rods with hysteresis. It is shown that such simulation can be accomplished by using circuits consisting of a finite number of loops with lumped constants. At least eight loops must be used to simulate a rod in cases without impact and more are necessary in cases with impact. Examples are given fo simulating forces acting on sections of an elastic-viscous rod and of an elastic rod with hysteresis with the $x=0$ end clamped and with a load on the $x=\tau$ end. *M. D. Friedman* (Newtonville, Mass.).

See also: Economics, Management Science: Vazsonyi. **Programming, Resource Allocation, Games:** Galler and Dwyer.

PROBABILITY

★ **Feller, William. An introduction to probability theory and its applications. Vol. I. 2nd ed.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957. xv+461 pp. $10.75.

The first edition of this book appeared in 1950 [MR 12, 424]. The second edition contains the same material but rearranged and with substantial additions. Especially the third chapter on fluctuations in coin tossing and random walks contains much new and gives a thorough discussion of the arc sine law and related questions. This law is illustrated graphically by the very striking result of a random experiment simulating 10,000 tosses of a coin. Also new is the twelfth chapter on compound distributions and branching processes, which contains a fuller discussion of these topics than the first edition.

Among many other changes some should be mentioned explicitly. The theory of recurrent events, which was stressed already in the first edition, is now given still greater dominance. The concept of waiting times is now introduced already in the second chapter. Bivariate generating functions are given a separate section. Processes involving escape (called degenerate processes in the first edition) are approached in a different way making such processes a non-pathological part of the general theory.

As in the first edition the exposition is mathematically rigorous and at the same time elegant and lucid. This fascinating book will remain a standard textbook of mathematical probability for many years to come.

U. Grenander (Providence, R.I.).

Thomasian, A. J. Metrics and norms on spaces of random variables. Ann. Math. Statist. **28** (1957), 512–514.

Proofs are furnished for the author's earlier announced results [C. R. Acad. Sci. Paris **242** (1956), 447–448; MR **17**, 864]. *S. C. Moy* (Detroit, Mich.).

Geffroy, Jean. Etude des diverses majorations asymptotiques des valeurs extrêmes d'un échantillon. C. R. Acad. Sci. Paris **244** (1957), 1712–1714.

This note announces ten theorems concerning the behavior of the minimum and the maximum in a series of n observed values of a random variable as a function of n. The theorem gives conditions under which a function, $B(n)$, is a majorant in probability, "almost surely" and "almost completely" of either of these two extremes.

C. C. Craig (Ann Arbor, Mich.).

Hagstroem, Karl-Gustav. Variables fondamentales du hasard. Giorn. Ist. Ital. Attuari **19** (1956), 84–91.

In an unlimited sequence of Bernoulli trials at each of which the probability of success is p the expectation of the number of trials required for first success is p^{-1}. Considering the bivariate probability distribution of the success ratio and the trials required for first success the author obtains, i.a., the regression equations and the correlation coefficient between the variables. [Other results may be obtained by using the methods of Ch. 3 of Frechet, Les probabilités associés à un système d'evénements compatibles et dépendants, t. I, II, Hermann, Paris, 1940, 1943; MR 3, 168; 7, 456.] *H. L. Seal.*

Rényi, A. A remark on the theorem of Simmons. Acta Sci. Math. Szeged **18** (1957), 21–22.

The author shows that the generalization of Simmons inequality, given by I. B. Haáz [same Acta **17** (1956), 41–44; MR **18**, 340] can also be derived from the original inequality of Simmons. *E. Lukacs* (Washington, D.C.).

Katz, Leo; and Powell, James H. Probability distributions of random variables associated with a structure of the sample space of sociometric investigations. Ann. Math. Statist. **28** (1957), 442–448.

The probability distributions examined are those associated with the directed linear graphs of n points used in social studies (a line from a to b is a line of communication, authority, liking etc., in this use). These graphs are subjected to various conditions on the lines, the most restrictive being that of t lines, r_i of which are directed toward, and s_i away from point i. The enumeration of such graphs (without consideration of topological equivalences) has been given previously by the authors [Proc. Amer. Math. Soc. **5** (1954), 621–626; MR **16**, 3]; an alternative proof is given here. This formula is used to determine the probability of k isolated points for fixed $p=(r_1, \cdots, r_n)$ or for fixed t, in each case in compressed symbolic form. *J. Riordan* (New York, N.Y.).

Cook, J. M. Rational formulae for the production of a spherically symmetric probability distribution. Math. Tables Aids Comput. **11** (1957), 81–82.

Let x_0, x_1, x_2, x_3 be distributed uniformly on $(-1, +1)$. Then

$$x=\frac{2(x_1x_3+x_0x_2)}{x_0^2+x_1^2+x_2^2+x_3^2}, \quad y=\frac{2(x_2x_3-x_0x_1)}{x_0^2+x_1^2+x_2^2+x_3^2},$$

$$z=\frac{x_0^2+x_3^2-x_1^2-x_2^2}{x_0^2+x_1^2+x_2^2+x_3^2},$$

with $x_0^2+x_1^2+x_2^2+x_3^2<1$ are the coordinates of a point uniformly distributed over the surface of the unit sphere. This result, which represents an extension of a method suggested by von Neumann [Nat. Bur. Standards Appl. Math. Ser. no. **12**, (1951) 36–38; MR **13**, 162] for producing a uniform distribution on the circumference of the unit circle, extends to n dimensions and to more general spaces. *G. E. Noether* (Tübingen).

Sanov, I. N. On the probability of large deviations of random magnitudes. Mat. Sb. N. S. **42** (84) (1957), 11–44. (Russian)

The subject of the title of the paper is developed in a series of theorems of which the following two are typical. Given an experiment having n mutually exclusive and exhaustive outcomes. Let p_i $(i=1, \cdots, n)$ be the probability of the ith outcome, ν_i, its relative frequency in N independent repetitions of the experiment. Let Ω be a closed set of points of the simplex $x_1+\cdots+x_n=1$, $x_i \geqq 0$, (satisfying some slight further condition specified in the paper). Then asymptotically

$$P\{(\nu_1, \cdots, \nu_n) \in \Omega\}=\exp N[\sum_{i=1}^{n} \lambda_i \ln(p_i/\lambda_i)+o(1)],$$

where $(\lambda_1, \cdots, \lambda_n)$ denotes the point in Ω at which $\sum_{i=1}^{n} x_i \ln(p_i/x_i)$ obtains its maximum, $(x_1, \cdots, x_n) \in \Omega$, (Theorem 2).

Let $F_N(x)$ be the empirical distribution function of a random sample of size N from a population with distribution function $F(x)$. Let $\Phi(x)$ be another distribution function such that $\int_{-\infty}^{\infty} \ln(dF/d\Phi)d\Phi$ exists. Then

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\lim_{N\to\infty} N^{-1}[\ln P\{\sup_x|F_N(x)-\Phi(x)|<\varepsilon\}] = \int_{-\infty}^{\infty}\ln(dF/d\Phi)d\Phi,$$

(Theorem 13).

The paper concludes with some indications of the applicability of the given theorems to the solution of known problems. *G. E. Noether* (Tübingen).

Dvoretzky, A. On a problem of Nelder and Hammersley. Bull. Res. Council Israel. Sect. A. **6** (1957), 115–118.

The principal results of this paper deal with the characterization of functions $F(x)$ which maximize

$$I(F)=\int_0^\infty\int_0^\infty \phi(x,y)dF(x)dF(y)$$

in the class of distribution functions, (i.e., monotone non-decreasing and of total variation one on $0\leqq x<\infty$), where $\phi(x, y)$ is a real, non-negative symmetric continuous function on $x\geqq 0$, $y\geqq 0$ with $\phi(x, y)\to 0$ as $x^2+y^2\to\infty$. A direct variational method is used to establish the existence of maximizing $F(x)$, and the property that the spectrum of any maximizing $F(x)$ is contained in the set of values x for which the corresponding function $\int_0^\infty \phi(x, y)dF(y)$ attains its maximum. Theorem I states that if in addition $\phi(x, y)$ has partial derivatives on $x\geqq 0$, $y\geqq 0$, and there is a function $g(x)$ such that $g(x)(\partial/\partial x)\phi(x, y)$ is a (strictly) convex function of x for each y then the distribution functions $F(x)$ for which $I(F)$ attains its maximum value are step-functions having at most two saltuses, and if there are two then one occurs at $x=0$. It is to be commented that the author's assumption of the mere existence of the partial derivatives of ϕ is not strong enough to justify certain steps of his proof; in particular, however, the details are valid if $(\partial/\partial x)\phi(x, y)$ is continuous in (x, y) on $x\geqq 0$, $y\geqq 0$. For the special instance of $\phi(x, y)=(x-y)^2e^{-x-y}$ Theorem I provides a simplified proof of an inequality of Hammersley [Compositio Math. **10** (1952), 241–244; MR **14**, 769] and also a proof of the uniqueness of the maximizing $F(x)$ in this case. Theorem II of the paper presents a characterization of the maximizing distribution functions in case $\phi(x, y)\to-\infty$ as $x^2+y^2\to\infty$ and there is a function $g(x)$ such that $g(x)(\partial/\partial x)\phi(x, y)$ is a polynomial in x. *W. T. Reid.*

Rustagi, Jagdish Sharan. On minimizing and maximizing a certain integral with statistical applications. Ann. Math. Statist. **28** (1957), 309–328.

The author is concerned with the variational problem of minimizing or maximizing $\int\varphi(x, F(x))dx$ subject to side conditions on the function $F(x)$. This function F is restricted to be a cumulative distribution function (c.d.f.) with specified first and second moments. The solution exists and is characterized when the distribution by F is restricted to a finite range. If $\phi(x, y)$ is strictly convex in y, the minimizing c.d.f. is unique and the maximizing c.d.f. is at most a three point distribution. The convexity of the space of c.d.f.'s and the space of attainable points whose coordinates are $\int\varphi(x, F(x))dx$, $\int xF(x)dx$, $\int x^2F(x)dx$ are used in the arguments. The ideas of this paper can be generalized in several directions. *H. Chernoff.*

Tumanyan, S. H. Asymptotic distribution of χ^2 criterion when the size of observations and the number of groups simultaneously increase. Teor. Veroyatnost. i Primenen. **1** (1956), 131–145. (Russian. English summary)

Let χ^2 be the χ^2-statistic based on n trials of an (s_n+1)-nomial distribution with probabilities p_{in}. Characteristic function expansions are used to prove that χ_n^2 is asymptotically $N(s_n, 2s_n)$ if s_n and all np_{in} approach ∞ with n. *J. Hannan* (East Lansing, Mich.).

Blackwell, David. On discrete variables whose sum is absolutely continuous. Ann. Math. Statist. **28** (1957), 520–521.

If $\{Z_n, n\geqq 1\}$, is any stochastic process with D states $0, 1, \cdots, D-1$ such that $X=\sum_1^\infty Z_nD^{-n}$ has an absolutely continuous distribution then the conditional distribution of $R_k=\sum_{n=1}^\infty Z_{k+n}D^{-n}$ given $Z_1, \cdots, Z_n$ converges to the uniform distribution on (0, 1) with probability one as $k\to\infty$. *K. L. Chung* (Syracuse, N.Y.).

Takács, L. On certain sojourn time problems in the theory of stochastic processes. Acta Math. Acad. Sci. Hungar. **8** (1957), 169–191.

Let $0=t_0<t_1<\cdots<t_n<\cdots$ be a sequence of randomly chosen time points; consider a system which is in state (I) in every time interval $t_{2n}\leqq t<t_{2n+1}$, and in state (II) in every interval $t_{2n+1}\leqq t<t_{2n}$. Suppose that the random variables $\xi_n=t_{2n+1}-t_{2n}$ and $\eta_n=t_{2n+2}-t_{2n+1}$ ($n=0, 1, 2, \cdots$) are independently and identically distributed and that all ξ_n [resp. all η_n] have a common distribution function $A(x)$ [resp. $B(x)$]. Let $\alpha(t)$ [resp. $\beta(t)$] be the total time which the system spends in (I) [resp. (II)] during the period $(0, t)$. The author expresses first the distribution function of $\beta(t)$ in terms of the distribution functions $A(x)$ and $B(x)$. He assumes then the existence of the first two moments of these distributions and shows that the suitably standardized random variables $\alpha(t)$ and $\beta(t)$ are asymptotically normal. He also derives asymptotic formulae for the probabilities $P_I(t)$ [$P_{II}(t)$] that the system is in state (I) [(II)] at time t as well as for the first two moments of $\beta(t)$. *E. Lukacs* (Washington, D.C.).

Rényi, A. On the asymptotic distribution of the sum of a random number of independent random variables. Acta Math. Acad. Sci. Hungar. **8** (1957), 193–199.

The author gives a simple proof of a particular case of a theorem of Anscombe [Proc. Cambridge Philos. Soc. **48** (1952), 600–607; MR **14**, 487] and uses it to derive a result of Takács concerning the asymptotic distribution of $\alpha(t)$. [For the notation and the statement of Takacs' result see the preceding review.] *E. Lukacs* (Washington, D.C.).

Prékopa, A.; and Rényi, A. On the independence in the limit of sums depending on the same sequence of independent random variables. Acta Math. Acad. Sci. Hungar. **7** (1956), 319–326. (Russian summary)

This paper contains results of the following type. Let $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nk_n}$ be a double sequence of stochastic variables, independent for each n. Form the sums $\zeta_l=\sum_k f_l(\xi_{nk})$, where $f_1(x), f_2(x), \cdots f_r(x)$ are integer-valued, Borel measurable functions such that $f_k(x)f_l(x)=0$ for $k\neq l$. If the random variables $f_l(\xi_{nk})$ are infinitesimal in probability and the variables ζ_l have a limit distribution as n tends to infinity, then the variables $\zeta_1, \zeta_2, \cdots, \zeta_r$ are asymptotically independent. This is applied to the study of the number of discontinuities of various magnitudes for a stochastic process with independent increments. *U. Grenander* (Providence, R.I.).

Prékopa, András. On the compound Poisson distribution. Acta Sci. Math. Szeged **18** (1957), 23–28.

A probability distribution is called a compound Pois-

son distribution if its characteristic function can be represented in the form

(*) $\phi(u)=$

$$\exp\left\{iyu+\int_{-\infty}^{0}(e^{iux}-1)dM(x)+\int_{0}^{\infty}(e^{iux}-1)dN(x)\right\},$$

where γ is a constant, $M(x)$ and $N(x)$ are defined on the intervals $(-\infty, 0)$ and $(0, \infty)$, respectively, both are monotone non-decreasing, $M(-\infty)=N(\infty)=0$, and further the integrals

$$\int_{-1}^{0}xdM(x), \int_{0}^{1}xdN(x)$$

exist. The author proves under certain conditions that (*) is the limit characteristic function of double sequences of independent and infinitesimal random variables and applies this theorem to stochastic processes with independent increments [see the paper reviewed above].

H. P. Edmundson (Santa Monica, Calif.).

Rogers, Hartley, Jr. A note on the law of large numbers. Proc. Amer. Math. Soc. **8** (1957), 518–520.

A counterexample is given to a theorem of Kolmogorov which has found its way into several recent textbooks. Namely, it is shown that in the weak law of large numbers with independent variables, the condition $n^{-2}\sum_1^n \mathrm{var}x_{nk}\to 0$ cannot be replaced by the condition $n^{-2}\sum_1^n Ex_{nk}^2\to 0$ where X_{nk} represents a variable X_k truncated at n. A similar counterexample was found independently by L. Breiman [Ann. Math. Statist. **28** (1957), 811–814]. A modification of this example is shown to give a stable sequence of variables with expectations which is not normally stable. *T. S. Ferguson* (Los Angeles, Calif.).

Motoo, Minoru. Some theorems on the sum of positive random variables. Ann. Inst. Statist. Math., Tokyo **7** (1956), 169–181.

Castoldi, Luigi. Apparenze conduttive e propagative in processi markoviani speciali. Processi conservativi e processi livellativi. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 156–164.

The author considers Markov processes and Markov chains, both in continuous time. For a chain let $f_i(t)$ denote the probability of state i at time t ($i=\cdots, -1, 0, 1, 2, \cdots$). Noting that $df_i/dt=\sum_j R_{ij}(t)f_j(t)$ ($i=\cdots, -1, 0, 1, 2, \cdots$), where $R_{ij}(t)$ is an appropriate matrix whose row-sums all vanish, the author calls the chain conservative if $\sum_i f_i(t)=1$ for every t, and "levelling" if

$$(d/dt)\sum_i\{f_i(t)\}^2=\sum_{i,j}f_iR_{ij}f_j\leqq 0.$$

He observes that certain rather stringent conditions (too complicated for reproduction here) are sufficient for a chain to be conservative. He illustrates these considerations by an application to birth-and-death chains, and illustrates his analogous developments for processes with a one-dimensional continuum of states by examples in which the probability density $f(x|t)$ satisfies differential equations of the "conduction" or "propagation" type.

H. P. Mulholland (Exeter).

Husu, A. P. On some functionals on random fields. Vestnik Leningrad. Univ. **12** (1957), no. 1, 37–45, 208. (Russian. English summary)

$u(x, y)$ is a Gaussian stochastic process over the parameter space $\{(x, y)|0\leqq x, y\leqq T\}$, $Eu(x, y)=0$. The expected values of a number of functionals of $u(x, y)$ are computed by evaluating multiple integrals. The functionals considered are $H_{ck}^2=T^{-2}\int_0^T\int_0^T u^2dxdy$, $H_{ca}=T^{-2}\int_0^T\int_0^T|u|dxdy$, $S(v)=$area of the pieces cut out by the plane $z=v$ from the surface $z=u(x, y)$, and $Q(v)=$volume between the surface $z=u(x, y)$ and the plane $z=v$. The author mentions the application of the stochastic process to the description of rough physical surfaces. *E. Reich.*

Patil, V. T. The consistency and adequacy of the Poisson-Markoff model for density fluctuations. Biometrika **44** (1957), 43–56.

Let R_j ($j=1, 2, \cdots, k$) and R^* be $k+1$ non-overlapping sub-divisions of an infinite space and let N_{tr} be the number of particles in region R_r at time t. Then the assumption (used in certain spermatozoa studies) that the stochastic motion of the particles is an emigration-immigration process [M. S. Bartlett, An introduction to stochastic processes, Cambridge, 1955; MR **16**, 939] for all modes of division of the space is shown to be contradictory. Numerical illustrations of the failure of the model are provided. *H. L. Seal* (New York, N.Y.).

Breiman, Leo. On transient Markov chains with application to the uniqueness problem for Markov processes. Ann. Math. Statist. **28** (1957), 499–503.

Consider a temporally homogeneous Markov chain $\{x_n, n\geqq 0\}$ with transition matrix $(p_{ij}, i, j\in I)$. A transient set of states C is denumerably atomic if $P(x_n\in C \text{ i.o.})>0$ and if for every infinite set $A\subset C$ we have $x_n\in C$ i.o. implies $x_n\in A$ i.o. with probability one. If $P(x_n\in C \text{ i.o.})>0$, C is denumerably atomic if and only if any bounded solution $\varphi(i)$ of $\varphi(i)\leqq\sum_{j\in I}p_{ij}\varphi(j)$ satisfy $\liminf_{i\in C}\varphi(i)=\liminf_{i\in C}\varphi(i)$. This is applied to the convergence of series on C and simple cases of Feller's uniqueness criterion [Trans. Amer. Math. Soc. **48** (1940), 488–515; MR **2**, 101] for the minimal solution to the Kolmogorov differential equations. *K. L. Chung* (Syracuse, N.Y.).

Blumenthal, R. M. An extended Markov property. Trans. Amer. Math. Soc. **85** (1957), 52–72.

Let B be a Borel field in a metric space. A Markov process $\{x(t), \mathfrak{F}(t), t\geqq 0\}$ where $\mathfrak{F}(t)$ is a nondecreasing family of Borel fields is a stochastic process such that (a) for each t, $x(t)$ is $\mathfrak{F}(t)$ measurable; (b) for each $s<t$ and $A\in B$, $P\{x(t)\in A|\mathfrak{F}(s)\}=P\{x(t)\in A|x(s)\}=P(t, x(s), A)$ where the function $P(t, \cdots)$ is subject to the usual measurability assumptions. A positive random variable T is a stopping time for the process if for each a, $\{w: T(w)<a\}\in\mathfrak{F}(a)$. Let $x'(t, w)=x(t+T(w), w)$, $t\geqq 0$ on $\Omega'=\{w: T(w)<\infty\}$; and let $\mathfrak{G}'(t)$ be the Borel field generated by $x'(s)$, $0\leqq s\leqq t$. A r.v. g depends on the M.P. only up to the time T if whenever $\{w: T(w)\leqq a\}\in\mathfrak{F}(b)$, $a\leqq b$ then $\{w: g(w)\in D; T(w)\leqq a\}\in\mathfrak{F}(b)$. A positive r. v. T is a Markov time if $\{x'(t), \mathfrak{G}'(t)\}$ is a M. P. with the same transition probability function as $\{x(t), \mathfrak{F}(t)\}$ and if for any g depending on the latter only up to T, g and $\{x'(t)\}$ are conditionally independent relative to $x'(0)$. {Reviewer's note: unless $x'(0)$ is assumed to be finite with probability one (say in a Euclidean space) the last stipulation requires clarification.}

Theorem. Let T be a stopping time. Suppose that (1) $P\{\lim_{s\downarrow 0}x'(t+s)=x'(t)|\Omega'\}=1$ for each t; (2) $\int\phi(y)P(t, x, dy)$ is continuous in x whenever ϕ is bounded and continuous; then T is a Markov time. The proof proceeds as follows: approximate T by a sequence of discrete r. v.'s decreasing to T. Each of these is a stopping time and since it is discrete it is easily shown to be a Markov time. The assumptions (1) and (2) permit the desired passage to the

limit. Similar ideas were used by G. A. Hunt in processess with independent increments [same Trans. **81** (1956), 294–319; MR **18**, 77]. Variations of the above theorem are given based on the regularity of sample functions, for which results similar to those by Kinney [ibid. **74** (1953), 280–302; MR **14**, 772] are proved. Further, if T_n is a sequence of stopping times tending to T with pr. 1 conditions are found under which $x(T_n)$ tends to $x(T)$ with pr. 1. Finally, conditions are found under which the stopped process $y(t, w)=x(\min(t, T(w)), w)$, where T is the first passage time to a given set, will be a M.P. *K. L. Chung.*

Dynkin, E.; and Jushkevich, A. Strong Markov processes. Teor. Veroyatnost. i Primenen. **1** (1956), 149–155. (Russian. English summary)

The definitions and the main result (the theorem quoted in the preceding review) are very close to those of the paper reviewed above, except a "stopping time" T there is called a "random variable independent of the future" here and the condition $\{w: T(w)<a\} \in F(a)$ is replaced by the slightly stronger one that $\{w: T(w)\leqq a\} \in F(a)$, in the notations of the preceding review. The same vagueness about $x(T(w), w)$ obtains when this value lies in the compactified space. The footnote on p. 151 is most interesting to this reviewer, unfortunately the cited paper by Jushkevich seems unavailable. In this footnote it is stated that in the denumerable case if an appropriate version is chosen, the strong Markov property is implied by the usual one, provided that $x(T(w), w)\neq\infty$ with probability one. This proviso excludes interesting applications. The point to prove is then, without this finiteness proviso, we have always $x(T(w)+t), w\neq\infty$ with probability one, for each fixed $t>0$. If this is proved then the cited result can be applied to the stopping time $T(w)+t$ for each $t>0$, and a useful form of the strong Markov property ensues.

K. L. Chung (Syracuse, N.Y.).

Dynkin, E. B. Markov processes and semi-groups of operators. Teor. Veroyatnost. i Primenen. **1** (1956), 25–37. (Russian. English summary)

"In this review the relations between various semi-groups of operators and between various infinitesimal operators connected with a homogeneous in t Markov process are investigated. General conditions are established under which the Markov process is determined by the corresponding infinitesimal operator. Let U_t be a semi-group of linear operators in the Banach space L such that $\|U_t\|\leqq 1$. Let $T_t=U_t^*$ be an adjoint semi-group in the conjugate space $B=L^*$. More abstractly the main object of this paper can be characterized as the study of semi-groups T_t and their infinitesimal operators in strong and weak topologies of space B" [the author's summary].

K. L. Chung (Syracuse, N.Y.).

★**Levy, Paul. Processus semi-markoviens.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 416–426. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

In a discrete parameter Markov chain the so-called sojourn time at a given state is distributed as q^n, in a continuous parameter as e^{-qt}. Both have the characteristic property that the length of further sojourn is independent of the length of what already has been. It is this property that made the process Markovian. Now if the probability laws of succession of states are unchanged but for each state an arbitrary sojourn distribution is assigned, we get a new kind of process which includes both the two types of Markov chains above and also the renewal process. This idea of the reviewer's was further generalized to allow the sojourn distribution to depend not only on the initial state but also the subsequent state. [A similar definition was given independently and simultaneously by W. L. Smith, Proc. Roy. Soc. London. Ser. A. **232** (1955), 6–31; MR **15**, 502.] The resulting process is called semi-Markovian. It is shown how the author's results on Markov chains [Ann. Sci. Ecole Norm. Sup. (3) **68** (1951), 327–381; **69** (1952), 203–212; MR **13**, 959; **14**, 663] can be extended to semi-Markovian processes, of which the state space is no longer assumed to be denumerable. Because of the arbitrariness of the sojourn distributions certain results (e.g. the ratio ergodic theorem) are no longer valid while others assume more general forms. The role of instantaneous states is discussed in some detail.

K. L. Chung (Syracuse, N.Y.).

Kendall, David G.; and Reuter, G. E. H. The calculation of the ergodic projection for Markov chains and processes with a countable infinity of states. Acta Math. **97** (1957), 103–144.

This memoir contains a very thorough and lucid discussion of the ergodic properties of temporally homogeneous Markov chains and processes with a countably infinite set of states. The work is crystallized into what the authors call problems A, B and C.

Problem A: a Markov chain is specified by the matrix $P=\{p_{ij}; i, j=0, 1, 2, \cdots\}$ of one-step transition probabilities. It is required to compute the "ergodic projection" $\Pi=(\pi_{ij})=\lim_{n\to\infty} n^{-1}(P+\cdots+P^n)$ (which is known to exist). The authors present a solution which does not involve a limiting process but may be simply expressed in terms of the non-negative solutions of $x_j=\sum_\alpha x_\alpha p_{\alpha j}$ and $y_i=\sum_\alpha p_{i\alpha} y_\alpha$ subject to the conditions that $\sum x_\alpha<\infty$ and $\sup_\alpha y_\alpha<\infty$, that is, in terms of the non-negative elements in the null-spaces $\mathscr{N}(\Delta)$ and $\mathscr{N}(\Delta^*)$ of the operators $\Delta=P-I$ and $\Delta^*=P^*-I$, which are taken to act on the Banach spaces l and m respectively.

Problem B: let $P_t=(p_{ij}(t))$ be the matrix of transition probabilities for a Markov process subject to the continuity condition $p_{ij}(t)\to\delta_{ij}$ as $t\downarrow 0$. It is required to find the "ergodic projection" $\Pi=\lim_{t\to\infty} P_t$ (which is known to exist). A one-parameter semi-group on l may be associated with P_t, and its infinitesimal generator Ω is defined by $\Omega\dot{x}=\text{strong lim}(P_t x-x)/t$. The matrix Π can then be simply expressed in terms of the non-negative elements of the null spaces $\mathscr{N}(\Omega)$ and $\mathscr{N}(\Omega^*)$.

Problem C: Let $Q=(q_{ij})=\lim_{t\downarrow 0}(P_t-I)/t$. In many cases of practical interest, the matrix Q turns out to be conservative, i.e. all its elements are finite and $\sum_\alpha q_{i\alpha}=0$ for each i. Conversely, given a conservative matrix Q, there exists at least one process for which $p_{ij}'(+0)=q_{ij}$. When this process is unique, Q is said to be regular and the corresponding process is called the Feller process determined by Q. The problem is then to determine Π_F, the Π-matrix of the Feller process, when a regular conservative Q is given. The solution is analogous to the solutions of the other two problems. It involves the positive solutions of $xQ=0$ where $x\in l$ and of $Qy=0$ where $y\in m$.

The general theory is illustrated by a few simple but illuminating examples.

In the second half of the paper the authors return to Problem B in a modified form, the new approach being made in the spirit of the theory of topological vector spaces. Since Π is an idempotent operator on l, it gives rise to a direct decomposition $l=\mathscr{R}(\Pi)\oplus\mathscr{N}(\pi)$, where

$\mathscr{R}(\Pi)$ is the range of Π. This suggests Problem B_1: to determine $\mathscr{R}(\Pi)$ and $\mathscr{N}(\Pi)$ when Ω is given. It is shown that $\mathscr{R}(\Pi)=\mathscr{N}(\Omega)$, but the identification of $\mathscr{N}(\Pi)$ causes serious difficulties. In general, $\overline{\mathscr{R}(\Omega)} \subseteq \mathscr{N}(\Pi) \subseteq [\mathscr{R}(\Omega)]_{c_0}$, that is, $\mathscr{N}(\Pi)$ lies between the strong closure of $\mathscr{R}(\Omega)$ and its c_0-weak closure. There are processes in which the inclusion is strict. *W. Ledermann* (Manchester).

Diederich, Franklin W. Expected number of maxima and minima of a stationary random process with non-Gaussian frequency distribution. NACA Tech. Note no. 3718 (1957), 21 pp.

S. O. Rice [Quart. Appl. Math. 12 (1955), 375–381; MR 16, 600] has given an estimation of the number of maxima of a stationary random process, when the function and its derivative have a normal (Gaussian) joint distribution. The present paper is concerned with a stationary random process, such that the function and its two first derivatives have a joint distribution differing slightly from the normal. The expected number of maxima or minima is expressed in terms of the second and higher moments of the distribution by means of a multivariate form of Edgeworth series. This approach furnishes essentially a correction to the results obtained for a Gaussian process; explicit expressions are given for the 2 first correction terms, involving the 3rd and 4th moments.

J. Kampé de Fériet (Lille).

Pinsker, M. S. Extrapolation of homogeneous stochastic fields and amount of information about a Gaussian field contained in another Gaussian stochastic field. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 815–818. (Russian)

The author treats prediction problems for a stationary process $\{\xi(t)\}$, where t: $(t_1, \cdots, t_n)$ is a point with integer-valued coordinates in n-space. Let $\sigma_{\xi\xi}{}^2$ be the mean square norm of the difference between $\xi(0)$ and the projection of this random variable on the closed linear manifold spanned by $\xi(t)$ for $t_1 \geqq 0$, $\sum_1^n |t_j| > 0$. A classification of processes is set up which makes the process singular if and only if $\sigma_{\xi\xi}{}^2=0$. In the non-singular case, generalizing the known result for $n=1$, it is found, applying a theorem of Czyan [same Dokl. 112 (1957), 207–210; MR 19, 327] who studied slightly different prediction problems for these processes, that

$$\sigma_{\xi\xi}{}^2 = (2\pi)^n \exp\left\{(2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int \log f_{\xi\xi}(\lambda_1, \cdots, \lambda_n) d\lambda_1, \cdots, d\lambda_n\right\}$$

where $f_{\xi\xi}$ is the spectral density. Several other results from the case $n=1$, including the Wold decomposition, are extended to this case. If two such processes, one of which is non-singular, are jointly Gaussian and stationary, the amount of information per step of one relative to the other is given by

$$\tfrac{1}{2}(2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int \log \frac{f_{\xi\xi} f_{\eta\eta}}{f_{\xi\xi} f_{\eta\eta} - |f_{\xi\eta}|^2} \, d\lambda_1 \cdots d\lambda_n,$$

where $f_{\xi\xi}$, $f_{\eta\eta}$ are the spectral densities of the processes and $f_{\xi\eta}$ is their mutual density. An analogous evaluation is made in the continuous parameter case. These evaluations were obtained previously by the author when $n=1$ [same Dokl. 99 (1954), 213–216; MR 16, 495].

J. L. Doob (Urbana, Ill.).

Daboni, Luciano. Osservazioni a proposito del problema del rinnovamento. Giorn. Ist. Ital. Attuari 19 (1956), 44–62.

The author studies two slight modifications of renewal theory, reducible at once to the usual theory. For example, in one case an individual has a certain distribution of lifetime. When he dies he is replaced by another individual, with the same lifetime distribution, but at a random age instead of age 0 as is customary. The author also studies a continuous parameter Markov chain with n states obtained by supposing that there are $\leqq n$ individuals each of which has probability μh of dying in the time interval $(t, t+h)$, supposing also that if k individuals are alive at any time t there is probability $\alpha_{n-k}h$, with $\alpha_0=0$, that a new individual will be introduced in the time interval $(t, t+h)$, both probabilities as stated up to an error of higher order than h. *J. L. Doob.*

Montroll, Elliot W. Random walks in multidimensional spaces, especially on periodic lattices. J. Soc. Indust. Appl. Math. 4 (1956), 241–260.

The author forms the generating function of a random walk on a k-dimensional lattice, with steps between nearest neighbor points, subject to periodic boundary conditions. Generalizing a method used by Watson, the probabilites of return to the starting point for $k=1, 2, 3$ are expressed in terms of the complete elliptic integral of the second kind; an asymptotic formula is given for k large. The results are extended to body centered and face centered lattices. Various remarks are made about Pearson's walk, mean motions, and physical applications. The paper is related to an earlier one [Proc. 3rd Berkeley Symposium Math. Statist. and Probability, 1954–1955, vol. 3, Univ. of California Press, 1956, pp. 209–246; MR 18, 836]. *M. Loève* (Berkeley, Calif.).

Tanaka, Hiroshi. On limiting distributions for one-dimensional diffusion processes. Bull. Math. Statist. 7 (1957), 84–91.

The author considers a diffusion process with inaccessible boundaries. He shows that if the expected first passage time between any two points is finite, then a limiting distribution exists (which obeys an integrated form of the time-independent "forward" equation).

P. Whittle (Wellington).

Watanabe, Satosi. A study of ergodicity and redundancy based on intersymbol correlation of finite range. Trans. I.R.E. PGIT-4 (1954), 85–92.

The first part of the paper is devoted to an exposition of the concept of ergodicity, as applied to a sequence of discrete random variables $\{a_\nu\}$. The concept of redundancy is then discussed by the logarithmic entropy measure. The word 'correlation' is used synonymously with dependence.

E. Reich (Minneapolis, Minn.).

Buch, Kai Rander. On a special use of the Erlang methods in industry. Teleteknik 1 (1957), 76–80.

A brief discussion, with numerical examples, of the calculation of the probability that at time t there will be m machines idle out of a total of n machines attended by one operator. The main assumptions made are that machines break down independently and Poisson-wise, and that repair-times have a negative-exponential distribution. The paper is stated to be an exposition of part of some earlier work by C. Palm, and references to this work are given. *W. L. Smith* (Cambridge, England).

See also: Analytic Theory of Numbers: Erdös; Delange (3 articles). **Measure, Integration:** Takahashi; Prékopa; Fréchet. **Harmonic Functions, Convex Functions:** Epstein and Scheerer. **Approximations, Orthogonal Functions:** Arató and Rényi. **Numerical Methods:** Kitagawa. **Fluid Mechanics, Acoustics:** Kampé de Fériet. **Quantum Mechanics:** Bellman, Kalaba, and Wing. **Economics, Management Science:** McMillan and Riordan; Luchak. **Information and Communication Theory:** Wilkinson. **Control Systems:** Drenick.

STATISTICS

Mazzoni, Pacifico. Su una particolare rappresentazione analitica degli istogrammi. Giorn. Mat. Finanz. (4) 1 (1955), 131–138.

Some formulas are given to assist in fitting frequency functions of the form $cx^m(x+a)^{-n}$, by successive approximations to a combined with the method of moments for the other parameters. *R. Solow* (Cambridge, Mass.).

Ríos, S. Sur la notion d'estimateur consistant. Trabajos Estadist. 7 (1956), 287–293. (Spanish. French summary).

The author attempts to relate two "definitions" of consistent estimator, both due to R. A. Fisher. The first of these is: A statistic satisfies the criterion of consistency if when it is calculated from the whole population, it is equal to the desired parameter [see Fisher, Philos. Trans. Roy. Soc. A. **222** (1922), 309–368, p. 316]. The other is that as the sample size increases; the estimator converges in probability to the true value. The author is apparently unaware that modern mathematical statistics recognizes the second definition, and he attempts to reconcile the two. He restricts his attention to statistics which are functionals of the sample cumulative distribution function. This excludes many estimators from consideration. He calls a functional continuous if it is continuous in the topology of uniform convergence. His main results are that for continuous functionals, the two definitions are equivalent, and that one even has convergence with probability one from the Glivenko-Cantelli theorem.

H. Rubin (Eugene, Ore.).

Basu, D. A note on the multivariate extension of some theorems related to the univariate normal distribution. Sankhyā **17** (1956), 221–224.

This is an expository paper which presents some immediate consequences of the definition given for the multivariate normal distributions by Fréchet [Ann. Inst. H. Poincaré **12** (1951), 1–29; MR **12**, 839], which runs as follows: The random p-vector x is said to have the p-variate normal distribution if for every constant p-vector t the distribution of tx' is a univariate normal distribution.

J. H. Curtiss (Providence, R.I.).

Huzurbazar, V. S. Sufficient statistics and orthogonal parameters. Sankhyā **17** (1956), 217–220.

Using a result of Koopman concerning the most general form of a two-parameter distribution which admits jointly sufficient statistics [Trans. Amer. Math. Soc. **39** (1936), 399–409] the author shows that in this case the problem of finding parameters which are orthogonal in the sense of Jeffreys, "Theory of probability", [2nd ed., Oxford, 1948; for a review of the 1st ed. see MR **1**, 151] is solvable. He exhibits the general solution and illustrates it for the case of a Type III distribution. *J. H. Curtiss.*

Lilliefors, H. W. A hand-computation determination of kill probability for weapons having spherical lethal volume. Operations Res. **5** (1957), 416–421.

Approximation of the distribution of $\sum_1^3 a_i x_i^2$ when the x_i are independent and normal (0, 1). *G. Elfving.*

Wishart, John. An approximate formula for the cumulative z-distribution. Ann. Math. Statist. **28** (1957), 504–510.

"A straightforward expansion and integration of the frequency function for Fisher's z produces a formula for the probability that z is not exceeded, of which the successive terms decrease rapidly when n_1 and n_2 are large. It is given in terms of incomplete normal moment functions (or χ^2 probabilities), and as a polynomial in $zN^{\frac{1}{2}}$, where N is the harmonic mean of n_1 and n_2. This last form is identical with the inverted Cornish-Fisher expansion, originally deduced by quite different methods." (From the author's summary.) Obvious misprints: on page 507 replace $2^{\frac{1}{2}}\pi$ by $(2\pi)^{\frac{1}{2}}$, and at top of page 509, W by N.

L. A. Aroian (Culver City, Calif.).

Laha, R. G. On some characterization problems connected with linear structural relations. Ann. Math. Statist. **28** (1957), 405–414.

The author considers a set observable random variables $x_0, x_1, \cdots, x_n$, having the following linear structural setup:

$$x_0 = a_{01}\xi_1 + a_{02}\xi_2 + \cdots + a_{0p}\xi_p + \eta_0$$
$$x_1 = a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1p}\xi_p + \eta_1$$
$$x_n = a_{n1}\xi_1 + a_{n2}\xi_2 + \cdots + a_{np}\xi_p + \eta_n$$

where $\xi_1, \xi_2, \cdots, \xi_p$, are the latent variables and $\eta_0, \eta_1, \cdots, \eta_n$ the error variables of the linear structural relation and the a_{ij}'s are a certain fixed set of constants. It is assumed that (1) the conditional distribution of x_0 for fixed $x_1, x_2, \cdots, x_n$ exists, (2) the set of random variables $\xi_1, \xi_2, \cdots, \xi_p$ is distributed independently of the set $\eta_0, \eta_1, \cdots, \eta_n$, (3) the latent variables $\xi_1, \xi_2, \cdots, \xi_p$, are mutually independent proper random variables each having a finite expectation and finite variance σ_j^2 $(j=1, 2, \cdots, p)$, (4) the error variables $\eta_0, \eta_1, \cdots, \eta_n$ are mutually independent (not necessarily proper) random variables, each with finite expectation and finite variance δ_k^2 $(k=0, 1, 2, \cdots, u)$. For $p=1$ and $n \geqq 2$ the author shows that if $E(\eta_0 | \eta_1, \eta_2, \cdots, \eta_n) = \sum_{j=1}^n \beta_j' \eta_j$, the β_j''s being a set of constants, then the multiple regression of x_0 on $x_1, x_2, \cdots, x_n$ is always linear, whenever the relation $a_{01} = \sum_{j=1}^n a_{j1}\beta_j'$ is satisfied. If in addition the variables $\eta_1, \eta_2, \cdots, \eta_n$ are mutually independent and $a_{01} \neq \sum_{j=1}^n a_{j1}\beta_j'$ then the necessary and sufficient condition for the multiple regression of x_0 on $x_1, x_2, \cdots, x_n$ to be linear is that ξ_1 and each of $\eta_1, \eta_2, \cdots, \eta_n$ is normally distributed. For the general case, if the constants a_{ij} are subject to the following restrictions (i) the vector $\alpha_j = (a_{1j}, a_{2j}, \cdots, a_{nj})$ has at least one non-zero element for each $j=1, 2, \cdots, p$, (ii) the matrix $A\Sigma A' + \Delta$ is non-singular, where $A = (a_{ij})$ $(i=1, 2, \cdots, n;\ j=1, 2, \cdots, p)$, A' is the transpose of A, and Σ and Δ are the dispersion matrices of $\xi_1, \xi_2, \cdots, \xi_p$ and $\eta_1, \eta_2, \cdots, \eta_n$ respectively (iii) each of the elements of the vectors $\alpha_0 \Sigma A'(A\Sigma A' + \Delta)^{-1}$ and $\alpha_0(I - \Sigma A'(A\Sigma A' + \Delta)^{-1}A)$ is different from zero, where $\alpha_0 = (a_{01}, a_{02}, \cdots, a_{0p})$, then the necessary and sufficient condition for $E(x_0 | x_1, x_2, \cdots, x_n) = \beta_1 x_1 + \beta_2 x_2 +$

$\cdots+\beta_n x_n$, and $V(x_0|\, x_1, x_2, \cdots, x_n)=\sigma_0^2$, is that each of $\xi_1, \xi_2, \cdots, \xi_p$ and each of the proper random variables amongst $\eta_1, \eta_2, \cdots, \eta_n$ are normally distributed.

S. Kullback (Washington, D.C.).

Harley, B. I. Relation between the distributions of non-central t and of a transformed correlation coefficient. Biometrika **44** (1957), 219–224.

If r is the correlation coefficient in a random sample of size n from a bivariate normal distribution with zero correlation, then $(n-2)^{\frac{1}{2}}r/(1-r^2)^{\frac{1}{2}}$ is known to be distributed exactly as Student's t with $(n-2)$ degrees of freedom. By a comparison of the first few moments the author has shown that if the population correlation, ρ, is not zero, then $v=(n-2)^{\frac{1}{2}}rg(\rho)/(1-r^2)^{\frac{1}{2}}$, where

$$g(\rho)=[2(1-\rho^2)/(2-\rho^2)]^{\frac{1}{2}},$$

is distributed approximately as non-central t, with $(n-2)$ degrees of freedom and $\delta=[(2n-3)\rho^2/(2-\rho^2)]^{\frac{1}{2}}$ as the non-central parameter. Some numerical comparisons are made and a method is given whereby the approximate equivalence of the distribution may be used to supplement the tables of the non-central t distribution given by Johnson and Welch [Biometrika **31** (1940), 362–389; MR **1**, 346] by using the more extensive tables of the probability distribution of r given by F. N. David [Tables of the ordinates and probability integral of the distribution of the correlation coefficient in small samples, Cambridge, 1938]. *Om P. Aggarwal* (Edmonton, Alta.).

Hudimoto, Hirosi. On the distribution-free classification of an individual into one of two groups. Ann. Inst. Statist. Math., Tokyo **8** (1956), 105–112.

Let π denote a composite population with proportions p and q $(p+q=1)$ of the populations π_1, π_2 with unknown continuous distribution functions F_1, F_2. If a sample value taken from π is classified into π_1 or π_2 according as its value is or is not less than x, then the probability of correct classification is

$$c(x)=pF_1(x)+q(1-F_2(x)).$$

If now there is available from π a sample of size N, for which it is also known that m elements are from π_1, and $N-m$ from π_2 then there exists an estimate $\widehat{c(x)}$ of $c(x)$, viz.

$$\widehat{c(x)}=\frac{m}{N}F_{m1}(x)+\frac{N-m}{N}[1-F_{N-m,2}(x)],$$

where F_{m1}, $F_{N-m,2}$ are the empirical d.f. estimating F_1F_2 respectively.

The optimum division point x^* is implied to be that value of x which maximizes $\widehat{c(x)}$ and a vague procedure is suggested for finding this maximizing x. A probability inequality is given for this estimate $\hat{x}^*$ under various regularity conditions. *D. G. Chapman* (Seattle, Wash.).

Hellwig, Z. Determining linear regression parameters by means of the two point method. Zastos. Mat. **3** (1956), 66–81. (Polish. Russian and English summaries)

Let (X, Y) be a two-dimensional random variable with linear regression of Y on X, i.e. such that $E(Y|X=x)=\alpha_x[X-E(X)]+E(Y)$, let (X_i, Y_i) for $i=1, 2, \cdots, n$ be a sample of (X, Y), and $\bar{X}=(\sum_{i=1}^n X_i)/n$, $\bar{Y}=(\sum_{i=1}^n Y_i)/n$. To estimate α_y by the "two point method", all sample points with $X_i>\bar{X}$ are grouped in set I and all others in set II, the coordinates X_IY_I, of the center of gravity of I are computed, and the estimator $a_Y=(Y_I-\bar{Y})/(X_I-\bar{X})$ is used. The author proves that a_Y is an unbiased and consistent estimate with asymptotically normal distribution, and he studies the relative efficiency of a_Y (with regard to the least squares estimator). *Z. W. Birnbaum.*

Moore, P. G. The transformation of a truncated Poisson distribution. Skand. Aktuarietidskr. **39** (1956), 19–25.

It has been observed [Anscombe, Biometrika **35** (1948), 246–254; MR **10**, 465] that if X is a Poisson variable with probabilities $\Pr(X=j)=\lambda^j e^{-\lambda}/j!$ and λ sufficiently large $(\lambda>2$, say) then $Y=\sqrt{(X+3/8)}$ is approximately normally distributed with variance close to $(1+1/16\lambda^2)/4\approx\frac{1}{4}$. The author proposes a method for estimating λ from truncated or censored samples of X, by first considering the transformed variable Y and then applying techniques which have been developed for estimating parameters of normal populations from truncated or censored samples.

Z. W. Birnbaum (Seattle, Wash.).

Saitô, Kin-ichirô. Maximum-likelihood estimate of proportion using supplementary information. Bull. Math. Statist. **7** (1956), 11–17.

The (extended) maximum likelihood estimator of $p_{1.}=p_{11}+p_{12}$, based on the multinomial frequencies n_{ij} in n trials of a 2×2 model with known $p_{.1}$, is $p_{.1}n_{11}/n_{.1}+p_{.2}n_{12}/n_{.2}$ and has asymptotic variance $n^{-1}[p_{1.}p_{2.}-|p_{ij}|^2/(p_{.1}p_{.2})]$. *J. Hannan* (East Lansing, Mich.).

Wolfowitz, J. The minimum distance method. Ann. Math. Statist. **28** (1957), 75–88.

The author proves the following extension of certain of his earlier results [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 602–606; MR **16**, 55]: with a certain definition of distance between distributions, and if an assumption he labels the "Identification and Continuity Assumption" is satisfied, then an estimate convergent with probability one can be obtained by taking that estimate which leads to a distribution "closest" to the empirical distribution. He makes the important observation that the distance function is arbitrary and irrelevant to the proof. The proof is immediate and straightforward, and is essentially the same, apart from the use of a different distance function, as that of a somewhat more general theorem of the reviewer [Ann. Math. Statist. **27** (1956), 200–203; MR **17**, 869]. A discussion is given of the Identification and Continuity Assumption, and a detailed treatment is given of two examples. *H. Rubin* (Eugene, Ore.).

Ray, W. D. A proof that the sequential probability ratio test (S.P.R.T.) of the general linear hypothesis terminates with probability unity. Ann. Math. Statist. **28** (1957), 521–523.

A wide class of sequential tests for the general linear hypothesis (Model I), based on the likelihood ratio for the standard F statistic, was proposed by N. L. Johnson [same Ann. **24** (1953), 614–623; MR **15**, 638]. Johnson's proof that the suggested test procedure reaches a terminal decision with probability one contained a flaw. The paper under review provides a new proof for this result, a proof that is of the same nature as that used by H. T. David and the reviewer [ibid. **27** (1956), 797–805; MR **18**, 345] in connection with a sequential t test.

There seems to be a small gap in the author's proof, since it does not cover the single exceptional case in which the common limit of the acceptance and rejection numbers equals the asymptotic mean of the test statistic.

W. Kruskal (Chicago, Ill.).

Wilkinson, John W. An analysis of paired comparison designs with incomplete repetitions. Biometrika **44** (1957), 97–113.

M. G. Kendall [Biometrics **11** (1955), 43–62; MR **17**, 758] and R. C. Bose [Biometrika **43** (1956), 113–121; MR **17**, 1102] have constructed experimental designs involving paired comparisons which are symmetrical with respect to objects and judges and yet which do not require each judge to compare all possible pairs of objects. R. A. Bradley and M. E. Terry [Biometrika **39** (1952), 324–345; MR **17**, 56] constructed significance tests for use when each judge compares all possible pairs of objects; these tests were constructed in relation to a probability-model in which it is supposed that the $\frac{1}{2}t(t-1)v$ probabilities

$p_{ij}^{u}(u) \equiv \mathrm{pr}[\text{judge } u \text{ will vote for } i \text{ when } i \text{ is compared with } j]$ $(1 \leq u \leq v;\ 1 \leq i < j \leq t)$ can be expressed in terms of the tv quantities $\pi_i(u)$ as follows: $p_{ij}^{u}(u) = \pi_i(u)/[\pi_i(u)+\pi_j(u)]$. R. C. Bose suggested to the author that he should extend the Bradley-Terry work to the Kendall-Bose designs, and in this paper the author constructs a number of likelihood-ratio tests based on the Bradley-Terry hypothesis and symmetrical paired-comparison designs with incomplete repetitions, the aim of which is to detect various modes of dependence of the π's on the object being judged and on the judge making the comparison. The tests are then applied to an experiment involving the pairwise comparison of handwriting specimens. *D. G. Kendall.*

Geary, R. C. Tests de la normalité. Ann. Inst. H. Poincaré **15** (1956), 35–65.

This is a valuable expository article which surveys, mostly without proofs, the author's earlier work. He first discusses the effect of non-normality of the universe on certain standard tests. A large part of the paper is devoted to the study of the following two families of test statistics:

$$a(c) = \Big(\sum_{j=1}^{n} |x_j - \bar{x}|^c\Big)/ns^c \tag{1}$$

and

$$g(c) = (-\Sigma' |x_j - \bar{x}|^c + \Sigma'' |x_j - \bar{x}|^c)/ns^c. \tag{2}$$

Here $\bar{x}$ is the sample mean and s is the sample standard deviation and the summation Σ' (Σ'') is extended over all observations x_j which are smaller than (greater than or equal to) $\bar{x}$. *E. Lukacs* (Washington, D.C.).

Richter, Hans. Parameterfreie Abschätzung und Realisierung von Erwartungswerten. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 147–162.

Let $t_1(x), \cdots, t_n(x), k(x)$ be $(n+1)$ measurable functions which are defined on a Borel set A of R_1. Consider a family $\mathfrak{F}$ of distribution functions such that (i) $\int_A dF(x) = 1$, (ii) the expectations of the $(n+1)$ functions $t_1, \cdots, t_n, k$ exist for all distribution functions of $\mathfrak{F}$. Let X be a random variable whose distribution function belongs to $\mathfrak{F}$ and let $\theta_1, \cdots, \theta_n, \varkappa$ be $n+1$ real numbers; the distribution function $F(x)$ is said to realize the $\theta_1, \cdots, \theta_n, \varkappa$ if $E[t_j(X)] = \theta_j$ $(j = 1, \cdots, n)$ and $E[k(X)] = \varkappa$. The author assumes that the expectations $\theta_1, \cdots, \theta_n$ of the $t_1(X), \cdots, t_n(X)$ are given. He investigates the problem of finding a function of the $\theta_1, \cdots, \theta_n$ which is an upper (lower) bound for $\varkappa = E[k(X)]$ where the expectation $E[k(X)]$ is determined for a distribution function which belongs to $\mathfrak{F}$ and which realizes $\theta_1, \cdots, \theta_n$. *E. Lukacs.*

Dwass, Meyer. On the distribution of ranks and of certain rank order statistics. Ann. Math. Statist. **28** (1957), 424–431.

Let $X_1, \cdots, X_N$ be N independent random variables, with respective ranks $R_1, \cdots, R_N$, where the distribution function of X_j is F_1 for $j \leq m$ and F_2 for $j > m$, both continuous. The generating function of the ranks is used to derive the joint limiting distribution of a fixed number of ratios R_j/N (which are asymptotically independent) as $N \to \infty$. These results are then used to show that a statistic of the form $\sum_{i=1}^{n} f_N(R_i/N)$ (whose asymptotic normality has been proved under various conditions) can have any stable asymptotic distribution as $m \to \infty$ and $n = N - m \to \infty$, provided that the functions F_1, F_2, f_N, and $n = n(m)$ are suitably chosen. *W. Hoeffding.*

Walsh, John E. Estimating future from past in life testing. Ann. Math. Statist. **28** (1957), 432–441.

Consider a life test which has been discontinued after one has obtained the first r failure times, $x_1 \leq x_2 \leq \cdots \leq x_r$. The author studies the problem of estimating θ_p, the $100p$ percent point of the underlying distribution; and x_R, the time when the Rth failure will occur, where $(n+1)p > r$ and $R > r$ respectively (i.e., estimation of population percentage points in the region not covered by the available data and predicting the future time at which the Rth failure will occur given the first r failures). Essentially the paper shows that non-parametric estimates can be given for θ_p and x_R for values of p and R which extend moderately into the region where sample data are not available. The proof of these results involves the expansion of θ_p and x_R in Taylor series and depends on previous work of F. N. David and N. L. Johnson [Biometrika **41** (1954), 228–240; MR **16**, 382]. *Benjamin Epstein.*

Derman, Cyrus. Stochastic approximation. Ann. Math. Statist. **27** (1956), 879–886.

Roy, J.; and Laha, R. G. On partially balanced linked block designs. Ann. Math. Statist. **28** (1957), 488–493.

The class of linked block (LB) designs has the characteristic that the number of treatments common to any two blocks is constant for all pairs of blocks. It is shown that the necessary and sufficient condition for a design D to be a LB design is that $k-\mu$ be a latent root of the matrix NN^T of multiplicity $b-1$, where $\mu = k(r-1)/(b-1)$, N is the incidence matrix of D, and k, r and b refer respectively to the number of plots per block, the number of replications per treatment and the number of blocks. Special attention is given to LB designs which are partially balanced with two associate classes. *W. S. Connor.*

Kramer, Clyde Young; and Bradley, Ralph Allan. Intra-block analysis for factorials in two-associate class group divisible designs. Ann. Math. Statist. **28** (1957), 349–361.

The paper shows how to use the group divisible incomplete block designs for two-factor experiments. The intra-block analysis is worked out, including estimates of the main and interaction effects, the analysis of variance, and t-tests. Many of the results in this paper are contained conjointly in papers by K. R. Nair and C. R. Rao [J. Roy. Statist. Soc. Ser. B. **10** (1948), 109–131; MR **10**, 554] and K. R. Nair [Calcutta Statist. Assoc. Bull. **5** (1953), 30–35; MR **15**, 494]. *W. S. Connor.*

Freeman, G. H. Some further methods of constructing regular group divisible incomplete block designs. Ann. Math. Statist. **28** (1957), 479–487.

The simplest of the author's methods is as follows. Divide mn varieties into m groups of n each. Form the blocks by choosing in all possible way alls varieties in u ($1 \leq u \leq m-1$) groups and h ($1 \leq h \leq n-1$) varieties from another group. Usually the number of replications obtained in this way is too large for practical purposes but in many cases the author succeeds in selecting from these designs a fraction of the blocks which form a group divisible design. Various modifications of this method too complicated to describe in a review yield a substantial variety of new designs within the useful range $r \leq 10$. In addition to these the author obtains several new designs by addition of known designs. *H. B. Mann.*

Wilk, M. B.; and Kempthorne, Oscar. Non-additivities in a Latin square design. J. Amer. Statist. Assoc. **52** (1957), 218–236.

An extensive study is made of the Latin square design in which the particular rows, columns, and treatments in the design are samples from populations of rows, columns, and treatments. Expectations of the appropriate mean squares are obtained without making assumptions about additivity of rows, columns, or treatments. Also a comparison is made between the randomized block and the Latin square designs. The conclusions are too numerous to be included here. *M. Zelen* (Washington, D.C.).

Lafon, Monique. Construction de blocs incomplets partiellement équilibrés à $s+1$ classes associées. C. R. Acad. Sci. Paris **244** (1957), 1714–1717.

The author shows how to construct a partially balanced incomplete block design corresponding to any pair of positive integers s, n not less than 2. The design has $s+1$ associated classes and parameters $v=2n^s$, $b=2n^{s-1}$, $k=2n$, $r=2$, $\lambda_1=2$, $n_1=1$, $\lambda_2=1$, $n_2=4(n-1)$, $\lambda_3=0$, $n_3=2(n-1)^2$ plus, for $4 \leq i \leq s+1$, $\lambda_i=0$, $n_i=2n^{i-2}(n-1)$. To explain the construction would be to republish the note. *R. H. Bruck* (Madison, Wis.).

Lafon, Monique. Blocs incomplets partiellement équilibrés à deux classes associées avec quatre répétitions. C. R. Acad. Sci. Paris **244** (1957), 1875–1877.

A list. *R. H. Bruck* (Madison, Wis.).

Thompson, H. R. On a new class of partially balanced incomplete block designs. Calcutta Statist. Assoc. Bull. **6** (1956), 193–195.

For an odd prime p and a positive integer n, set $s=p^n$. The author constructs a partially balanced incomplete block design with parameters $v=b=2s$, $r=k=s$, $\lambda_1=(s-1)/2$, $n_1=2s-2$, $\lambda_2=s-1$, $n_2=1$ and

$$p^1=\begin{pmatrix} 2s-4 & 1 \\ 1 & 0 \end{pmatrix}, \quad p^2=\begin{pmatrix} 2s-2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The design is group divisible and may also be regarded as a partially balanced Youden square. *R. H. Bruck.*

Kempthorne, Oscar. The efficiency factor of an incomplete block design. Ann. Math. Statist. **27** (1956), 846–849.

Author's summary: It is shown that the efficiency factor of a design is r times the harmonic mean of the latent roots of the reduced intrablock normal equations excluding the root which is always zero. *R. H. Bruck.*

Thompson, W. A., Jr. A note on the balanced incomplete block designs. Ann. Math. Statist. **27** (1956), 842–846.

The author proves the following theorem. If the estimates of the treatment effects in an incomplete block design all have the same variances and the same covariances, then the design is a balanced incomplete block design. The proof is based upon the following lemma. If C is a real, symmetric $v \times v$ matrix all of whose row-sums are zero, and if C possesses a nonzero characteristic root e of multiplicity $v-1$, then $C=(e/v)S$ where S is the matrix with $v-1$ down the main diagonal and with -1 in all other positions. {Reviewer's note: The lemma is stated in the paper with the words "real, symmetric" omitted. In this form it is false for $v \geq 2$, as is easily seen in terms of integral triangular matrices}. *R. H. Bruck.*

Nair, K. R. Simplified analysis of singly linked blocks. Biometrics **12** (1956), 369–380.

The mathematical aspects of the paper are contained in the first four pages. After a discussion of singly linked block designs (these are duals of balanced incomplete block designs) the author presents and explains the formulas necessary for his form of the appropriate statistical analysis. Another seven pages of the paper (including roughly four pages of tabular detail) are devoted to an illustrative example based on an experiment conducted at the North Carolina Agricultural Experiment Station. The experiment involved 35 varieties of wheat arranged in 15 blocks of 7 plots according to the singly linked block design with parameters $v=35$, $b=15$, $k=7$, $r=3$, $\lambda_1=1$, $n_1=18$, $\lambda_2=0$, $n_2=16$. *R. H. Bruck* (Madison, Wis.).

Jensen, Arne. The applicability of decision theory in the planning and operation of telephone plant. Teleteknik **1** (1957), 126–129.

This paper is a largely informal account of the use of statistical decision theory in telephone traffic engineering. It is claimed that for most purposes Bayes solutions should be used, and that these can be obtained using a certain correction from Moe's Principle [Copenhagen Telephone Co., 1950; MR **12**, 624]. *V. E. Beneš.*

Reiter, Stanley. Surrogates for uncertain decision problems: minimal information for decision making. Econometrica **25** (1957), 339–345.

Let a general decision problem be expressed as that of choosing x so as to maximize $\int f(x, y)d\phi(y)$. Then, as the author stresses, it may, in practice, not be necessary to know much about the distribution of ϕ to determine one or all of the optimizing values of x. In particular, if $f(x, y)$ is of low rank, that is, if $f(x, y)=\sum A_i(x)B_i(y)$, where i has a small range, then ϕ clearly enters the problem only through the few moment-like quantities $\eta_i=\int B_i(y)d\phi(y)$. *L. J. Savage* (Chicago, Ill.).

Fraser, D. A. S. A regression analysis using the invariance method. Ann. Math. Statist. **28** (1957), 517–520.

Application of the invariance principle to the estimation of the coefficients in a linear regression model with rectangularly distributed error terms. The solution is not carried out explicitely; its idea applies to any error term distribution. *G. Elfving* (Helsinki).

Sobel, Milton; and Huyett, Marilyn J. Selecting the best one of several binomial populations. Bell System Tech. J. **36** (1957), 537–576.

From author's summary: The authors have prepared

tables for use in any experiment designed to select that particular one of k binomial populations with the highest probability of success. Before experimentation the experimenter chooses two constants d^* and p^* ($0<d^*\leq 1$; $0\leq p^*<1$) and specifies that he would like to guarantee a probability of at least p^* of a correct selection whenever the true difference between the probabilities of success associated with the best and second best populations is at least d^*. Separate tables are given for $k=2, 3, 4$, and 10 which show the smallest number of units required per population to satisfy this specification. Each table gives the result for $d^*=0.05$ (0.05) 0.50 and for $p^*=0.50$, 0.60, 0.70 (0.05) 0.95, and 0.99. Graphs are given on which interpolation can be carried out for values of d^* and p^* not considered in the tables. Graphs are also given which make possible an interpolation or extrapolation for other values of k. An alternative specification is given for use when the experimenter has some a priori knowledge of the probabilities of success. *S. Isaacson.*

Karlin, Samuel. Pólya type distributions. II. Ann. Math. Statist. **28** (1957), 281–308.

[For part I see Proc. 3rd Berkeley Symposium on Math. Statist. and Probability, 1954–1955, v. 1, Univ. of California Press, 1956, pp. 115–128; MR **18**, 947]. A distribution is of (strictly) Pólya type n, $2\leq n\leq\infty$, if there is a density p with respect to some measure such that $|p(x_i, \omega_j)|\geq 0$ (>0) if the x's and ω's each form an increasing sequence and the order of the determinant is at most n. Equivalent conditions are given for the distribution to be of Pólya type n. For $n=2$, this is the case of a monotone likelihood ratio, treated by the author and the reviewer [Ann. Math. Stat. **27** (1956), 272–299; MR **18**, 425]. The basic property of Pólya type n distributions is that if h has less than n sign changes $\int h(\omega)p(x, \omega)dF(\omega)$ has at most as many sign changes as h, and similarly with x and ω interchanged. For strictly type 3 problems, uniformly most powerful unbiased tests exist. Also the minimax problem in the two-action case is discussed in considerable detail. *H. Rubin* (Eugene, Ore.).

Sen, A. R. A simple design in sampling with varying probabilities. J. Indian Soc. Agric. Statist. **7** (1955), 57–69.

Theory has been developed for the unbiased system when $2n$ primary sampling units (p.s.u.s.) are selected from a stratum in two groups of n each. Simple expressions for the unbiased estimates of the total and of the variance of the estimated total are derived. The theory has been generalised for the selection of mn p.s.u.s. from a stratum in m groups with replacement with n p.s.u.s. in each group. *From the author's summary.*

Gautschi, Werner. Some remarks on systematic sampling. Ann. Math. Statist. **28** (1957), 385–394.

The precision of systematic sampling with multiple random starts is compared with that of ordinary systematic sampling with a single random start, the samples being of the same size in both cases.

For the purposes of comparison the sampled finite population of N values $y_1, \cdots, y_N$ is itself considered as a sample of an infinite higher class-population with distribution having some specified properties. Three types of population are considered: (1) populations in random order, (2) populations with linear trend, (3) populations with serial correlation [for definitions of these types see, e.g., Cochran, Sampling techniques, Wiley, New York, 1953; MR **14**, 887]. The precision of various types of sampling is then expressed in terms of the expected variance of the corresponding estimate, the expectation being taken over all possible samples. It is proved, that for populations in random order the two types of sampling are equally precise, whereas for populations with linear trend systematic sampling with a single random start is superior to that with multiple random starts, as far as precision is concerned. The main result of the paper is contained in the Theorem of Section 4, which states that if the correlogram of the population is convex, then systematic sampling with a single start is more precise than systematic sampling with multiple random starts. A short table of asymptotic relative precisions for the case of an exponential correlogram is attached. *J. Janko.*

Koźniewska, I. Comparison of the efficiency of drawing samples with and without replacement when the variance of the general population is unknown. Colloq. Math. **4** (1957), 232–238.

Let $x_1, \cdots, x_N$ be a finite population with mean μ and variance σ^2. If the sample $X_1, \cdots, X_n$ is obtained by drawings with replacements, consider the statistics

$$\bar{X}=\frac{1}{n}\sum_{j=1}^{n} X_j,\quad m_2=\frac{1}{n}\sum_{j=1}^{n}(X_j-\bar{X})^2,$$

$$M_2=\frac{n}{n-1}m_2,\quad a_2=\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2;$$

if the sample was obtained without replacements, let

$$m_2^*=\frac{1}{n}\sum_{j=1}^{n}(X_j-\bar{X})^2,\quad M_2^*=\frac{n}{n-1}\cdot\frac{N-1}{N}m_2^*,$$

$$a_2^*=\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2.$$

The author proves the following inequalities between the variances of unbiased estimates of σ^2: $\operatorname{var}(M_2^*)<\operatorname{var}(M_2)$, $\operatorname{var}(a_2^*)<\operatorname{var}(a_2)$; $\operatorname{var}(a_2)<\operatorname{var}(M_2)$; $\operatorname{var}(a_2^*)<\operatorname{var}(M_2^*)$, and $\operatorname{var}(M_2^*)<\operatorname{var}(a_2)$ if and only if the coefficient of excess μ_4/σ^4 of the population is greater than a certain function of N and n. *Z. W. Birnbaum.*

Brownlee, K. A. A note on the effects of nonresponse on surveys. J. Amer. Statist. Assoc. **52** (1957), 29–32.

A consideration of the bias introduced into a survey of the association between smoking and lung cancer if the ratio of recruitment rates to the survey of healthy and unhealthy persons differs for smokers and non-smokers. *P. Whittle* (Wellington).

★ **Putz, R. R. A method for the measurement of the correlation function and ordinate distributions for two time-history functions.** Institute of Engineering Research, Univ. of California, Berkeley, Calif., 1957. 39 pp.

Estimation of the cross-correlations and cross-spectrum of two time-series by use of signs (only) of deviations from mean values. *P. Whittle* (Wellington).

Allais, Maurice. Test de périodicité. Généralisation du test de Schuster au cas de séries temporelles autocorrélées. C. R. Acad. Sci. Paris **244** (1957), 2469–2471.

Let ξ_t be a discrete stochastic process with $\xi_t=\eta_t+\zeta_t$, $\zeta_t=\varepsilon_t+a_1\varepsilon_{t-1}+a_2\varepsilon_{t-2}\cdots a_h\varepsilon_{t-h}$ where ε_t are $ND(0, \sigma^2)$, η_t is a deterministic process. Let $\{x_t\}$ be an observed realization of the process and r_ν the sample auto correlation

coefficient of lag v. The author states that for large N

$$\Pr\left[\frac{NI_\omega}{4\sigma^2 k_\omega} > \mu\right] \approx e^{-\mu}$$

where

$$I_\omega = A_\omega^2 + B_\omega^2,\ A_\omega = \frac{2}{N}\sum_{v=0}^{N-1} x_{v+1}\cos v\omega$$

$$B_\omega = \frac{2}{N}\sum_{v=0}^{N-1} x_{v+1}\sin v\omega,$$

$$k_\omega = 1 + 2\sum_{v=1}^{h}\left(1-\frac{v}{N}\right)r_v\cos v\omega.$$

This is an extension of a classical result due to Schuster and will serve as a large sample test for the significance of "hidden periodicities" in autocorrelated time series. There are numerous misprints. *D. G. Chapman.*

McGill, William J. Multivariate information transmission. Trans. I.R.E. PGIT-4 (1954), 93–111.

In an experiment each individual is placed in one of a finite number of categories according to a variable y and simultaneously into categories according to variables u, v, $\cdots$. The author studies the dependence of y on $u, v, \cdots$. His approach is through the information transmitted from $u, v, \cdots$ to y, and leads to a series of likelihood ratio tests analogous to those usually carried out on the contingency table, but using a likelihood ratio criterion rather than χ^2. The same topic is also discussed in a more recent paper [W. R. Garner and W. J. McGill, Psychometrika 21 (1956), 219–228; MR 18, 367]. *D. V. Lindley.*

Guttman, Louis. Simple proofs of relations between the communality problem and multiple correlation. Psychometrika 22 (1957), 147–157.

Let x_j denote the jth observed variable and x_{ji} the score of the individual i on x_j with $\sum_i x_{ji}=0$ $(j=1, 2, \cdots, n)$. Then let each x_j be the sum of two uncorrelated components, y_j and z_j, i.e., $x_{ji}\equiv y_{ji}+z_{ji}$, such that $\sum_i y_{ji}z_{ji}=0$ $(j=1, 2, \cdots, n)$ and $\sum z_{ji}x_{ki}=0$ $(j\neq k;\ j, k=1, 2, \cdots, n)$. These conditions define what is here called the s-law of deviation which is studied without the use of matrix methods, deriving a number of more and less well known results in factor theory. It is shown that the z_j's may be either residuals in the fitted regression of x_j on the remaining $n-1$ observed variables or unique factors in the Spearman-Thurstone model. Inequalities and bounds are obtained involving regressions and communalities, some of which have a wider applicability. In conclusion, it is pointed out the s-law can be subsumed in a biorthogonality relation between the x's and the z's alone whereas in factor theory attention has customarily been centered on the y's. *C. C. Craig* (Ann Arbor, Mich.).

de Vries, M. Experiment and research in statistical economics. Statistica, Neerlandica 11 (1957), 77–87. (Dutch. English summary)

In the design of experiments for scientific purposes it is assumed that the sample elements do not interfere with the experiment and do not influence the observations. In economic research, however, these assumptions are not always fulfilled.

In this field of investigation the sample elements are for the major part human beings. It might be expected that human subjects will not submit passively to any experiment.

In this paper methods of observation and experimentation are discussed in which the human influence has been largely eliminated. Some topics of sample design have been elucidated by means of an example. *Author's summary.*

See also: Probability: Tumanyan. **Fluid Mechanics, Acoustics:** Phillips; Kampé de Fériet. **Optics, Electromagnetic Theory, Circuits:** Bose. **Astronomy:** Ramakrishnan (2 articles). **Economics, Management Science:** Hüsser; Müller; Knoche. **Biology and Sociology:** Smith.

PHYSICAL APPLICATIONS

★ **Finzi, Bruno. Teoria dei campi.** Libreria editrice politecnica Cesare Tamburini, Milano, 1954. ii+294 pp. 1800 Lire.

This volume is based on the author's lecture notes on mathematical physics given at the University of Milan in 1953–1954. The first chapter is devoted to vector fields in a three dimensional space, and contains many interesting physical examples. In the second chapter, the author first gives very good examples of tensors, and then develops the tensor algebra also in the three dimensional space. Then he discusses vectors and tensors in n-dimensional general spaces with metrics. This chapter gives a very concise summary of the theory of Riemannian spaces. Chapter III is devoted to the theory of deformations, and the following three chapters deal with fields employed in kinetics, theory of elasticity and theory of heat. Chapters VII and VIII are devoted respectively to gravitational and electromagnetic fields. In chapter IX, entitled "Electromagnetic fields in space-time", the author gives a very concise exposition on Lorentz transformations and shows how to transcribe the equations of Lorentz-Maxwell in four-dimensional forms. This chapter also contains a discussion of the energy tensor and the meson field. Chapter X contains Einstein's theory of gravitation. In Chapter XI, the author presents the so-called unitary field theory of Einstein based on a non-symmetric tensor field, and he shows how to derive the field equations in this theory. In Chapter XII, the author returns to fields appearing in analytical dynamics. The last chapter contains some remarks on fields in thermodynamics and statistical mechanics. *K. Yano* (Tokyo).

See also: Integral and Integrodifferential Equations: Mikhlin.

Mechanics of Particles and Systems

★ **Терских, В. П. [Terskih, V. P.] Метод цепных дробей в применении к исследованию колебаний механических систем. Том I. Простые линейные и нелинейные системы. [The method of continued fractions applied to the investigation of oscillations of mechanical systems. Vol. I. Simple linear and nonlinear systems.]** Gos. Soyuz. Izdat. Sudostroit. Promyšl., Leningrad, 1955. 375 pp. 20.60 rubles.

The main object of this book is to apply the method of continued fractions to the vibrating problems of chain systems (mechanical and electrical systems). The author gives a review of his own work and the work of other

authors on this subject. The topic is explained in two volumes with four parts. The present first volume has two parts. The first part deals with simple linear systems and has five chapters: introduction, free and forced vibrations without friction, linear systems with friction, and reductor-systems. The second part is devoted to the simple non-linear systems and has seven chapters: free and forced vibrations without friction, forced vibrations with high friction, systems with low friction, systems with non-symmetric elements, special cases of non-linear vibrations, and estimation of the error of approximate solutions. The proposed method is applied to the determination of the frequencies and modes of free vibrations, of the amplitudes and initial phases of forced vibrations, and of the influence of system parameters on the frequencies of free vibrations or on the amplitudes of forced vibrations.

The detailed presentation is clear and followed by very well drawn graphs. Unfortunately, there are no examples, but the author refers to his book "The calculation of the torsional vibrations···" [3 vols., Mašgiz, 1953, 1954].

There is a bibliography of over 39, mainly Russian, books and references. Paper and print are good. The book is of practical interest to mechanical engineers.

D. Rašković (Belgrade).

Freudenthal, H. The Foucault pendulum experiment in differential geometry. Simon Stevin **31** (1957), 49–60. (Dutch)

If the sphere T of the earth is taken at rest, and around T moves a rotating sphere S, and if at a point p of S coinciding with a point p_0 of T a linear oscillator is active in the tangent plane, then the plane of the oscillator moves on T according to the parallel displacement law of Levi-Civita and Schouten. This property is discussed with some illustrations concerning cone, sphere and non-euclidean geometry, and a historical account is given of the Foucault experiment. This experiment dates from 1851 and was performed in the Pantheon in Paris; it was such a world sensation that similar experiments were soon on exhibition in many cities, including Rio de Janeiro and Quebec in America. [C. S. Lyman has reported on Foucault experiments in the USA, performed at Boston, New Haven and Providence [Amer. J. Sci. Acts (2) **12** (1951), 251–255, 398–416]; this article also discusses the property that the direction of the pendulum plane, laid out in the developed tangent cone, remains parallel in the ordinary sense.]

D. J. Struik (Cambridge, Mass.).

Rund, Hanno. Über allgemeine nicht-holonome und dissipative dynamische Systeme. Schr. Forschungsinst. Math. **1** (1957), 269–279.

x^α ($\alpha=1, \cdots, n-1$) sind die Koordinaten eines mechanischen Systems; es wird der Raum F_n der Variabelen $x^i=x^\alpha$, t betrachtet und darin mittels der Lagrangeschen Funktion des Systems eine allgemeine Metrik definiert. In einer früheren Arbeit [Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 41–51; MR **15**, 659] hat Verf. auf ein geometrisches Prinzip hingewiesen: die Bahnkurve des Systems soll unter allen Vergleichskurven welche dieselbe Tangente in einem Punkte besitzen und denselben Bindungen unterworfen sind, die kleinste Krümmung haben. Das Prinzip wird jetzt auf nicht-holonome und dissipative Systeme angewandt (die Beschränkung auf eine einzige nicht-holonome und nicht-lineare Bindung ist nicht wesentlich) wobei sich zeigt dass man weit ausgehen kann über die Beschränkungen welche man üblicherweise der Lagrangeschen Funktion und den Bindungen auflegt. Die unbestimmten Multiplikatoren, die sonst in den nicht-holonomen Theorien auftreten, werden ausgeschaltet; auch darf die von den Reaktionen geleistete Arbeit von der Zeit und der Geschwindigkeit abhängen, so dass dissipative Systeme miteinbegriffen werden.

O. Bottema (Delft).

Rubin, Hanan; and Ungar, Peter. Motion under a strong constraining force. Comm. Pure Appl. Math. **10** (1957), 65–87.

A conservative holonomic non-linear dynamical system S_1 subject to certain smooth constraints is compared with a similar unconstrained system S_2 provided with extra conservative holonomic forces $\mu\vec{F}$ which are zero whenever S_2 satisfies the constraints of S_1 but tend strongly to restore S_2 to these constraints whenever deviation occurs. Intuition is to the effect that solutions of S_2 would tend to solutions of S_1 as $\mu\to\infty$, at least if S_2 satisfies the constraints initially. This turns out to be the case if the initial velocity vector is tangent to the constraints. Otherwise the situation is more complicated. The system S_1 is of lower order than S_2, so that the dependence of S_2 on the parameter μ is singular.

D. C. Lewis, Jr. (Baltimore, Md.).

Išlinskiĭ, A. Yu. On the equations of longitudinal motions of a cable (elastic cord) of variable length. Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 939–941. (Russian)

The article deals with the motion of a cable bearing a load on one end, while the other end lies over a turning wheel. If x is the distance of an element of the unstretched cable from the load and $u(x,t)$ is the displacement of the element relative to the load, then an integro-differential equation for $u(x,t)$ is found, in which the approximation $u(x,t)=x\varphi(t)$ is substituted.

W. H. Muller (Hague).

Savin, G. N.; Postol'nik, Yu. S.; and Ševelo, V. N. On the equation of the dynamics of an elastic or visco-elastic thread of variable length. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 36–40. (Ukrainian. Russian summary)

The authors are concerned with longitudinal vibrations of an elastic or visco-elastic thread which hangs over a cylinder revolving about a fixed horizontal axis with velocity $v(t)$, to whose lower end a load Q is attached. The position of the point $M(X)$ of the thread is determined by the relations $X=\xi(t)-x-u(x,t)$ and $\xi(t)=l(t)+u(l,t)$. Since the mass of the system is variable, assume that the relative velocity of the variable-mass particles is zero. Under this assumption the Lagrangian equations of motion of a system with variable mass have the same form as those of a system with constant mass. The authors restrict themselves to the case where the weight of the thread is small in comparison to the weight Q of the attached load. In this case, $u(x,t)=x\varphi(t)$ and the Lagrangian system reduces to a single second order non-linear differential equation for $\varphi(t)$

$$(1)\quad (l/g)(Q+\tfrac{1}{3}ql)\ddot{\varphi}+(l/g)(Q+\tfrac{1}{2}ql)l\dot{\varphi}+EF\varphi = (l/g)(Q+\tfrac{1}{2}ql)(g-dv/dt),$$

$$l=l_0+\int_0^t \frac{v}{1+\varphi}\,dt.$$

This equation, governing the vibrations of an elastic thread, was previously obtained by Išlinskiĭ [see the paper

reviewed above] using a different method. The corresponding equation [say (2)] for the vibrations of an viscoelastic thread is obtained from the above equation (1) by adding to its left hand side a dissipative force of the form $\alpha\dot{\varphi}(t)$ [cf. also Savin, Dokl. Akad. Nauk SSSR **97** (1954), 991–994; MR **16**, 533]. *E. Leimanis.*

Ziemba, Stefan. The influence of mass and internal friction on free torsional vibrations of a bar. Arch. Mech. Stos. **9** (1957), 51–72. (Polish and Russian summaries)

After an expository introduction, the author deduces the differential equations of torsional vibrations of a cylindrical bar with variable cross-section, taking into account the proper mass as well as the internal and external frictions, and assuming that the coefficient of internal friction is constant.

The differential equations of the restricted problem of the circular cylindrical bar with constant coefficient of external friction are solved by the technique of separation of variables for various boundary conditions.

The case of proper (eigen) vibrations of a circular cylinder with one end fixed and a coaxial disk rigidly connected to the other end is separately analyzed.

T. P. Anđelić (Belgrade).

Carstoiu, John. Sur le minimum du temps de montée d'un avion à réaction. C. R. Acad. Sci. Paris **244** (1957), 2285–2286.

See also: Differential Geometry: Gallissot. **Astronomy:** Klemperer and Baker.

Statistical Thermodynamics and Mechanics

★ **Гейликман, Б. Т. [Geĭlikman, B. T.] Статистическая теория фазовых превращений. [The statistical theory of phase transitions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 119 pp. 3.15 rubles.

The subject of this monograph is the theoretical investigation of a very important phase of statistical physics, namely, physics of transition of matter from one state into another (phase transitions). A study of phase transitions in systems composed of elements exerting a mutual reciprocal interaction upon one another presents major difficulties. Gellikman reviews several approaches toward a solution of this problem. In particular, he discusses the statistical theory of phase transitions of the first and second kinds. One of the problems is to find the proper expression for the sum of states of systems consisting of mutually interacting elements. As the result of his discussion, it is possible to derive the necessary conditions for obtaining the transitions of the first and second kinds and to explain the mechanism of these transitions. The character of this theory does not allow a more detailed investigation of numerous phenomena which arise in the course of the transitions. The particular sections of the monograph treat the following items: 1. Calculation of the number of states; 2. dependence of group integrals upon the boundary curve and their asymptotic behavior; 3. phase transitions of the first kind; 4. critics of the Mayer theory of condensation; 5. phase transitions of the second kind — systems composed of many elements; 6. phase transitions of the second kind for a system subjected to quantum statistical interactions; and 7. approximate calculation of the number of states for a system subjected to classical statistical interactions. *M. Z. v. Krzywoblocki* (Urbana, Ill.).

Marquet, Simone. Base de la théorie cinétique. Évolution et irréversibilité des systèmes. C. R. Acad. Sci. Paris **244** (1957), 1463–1466.

Un système de n corpuscules est supposé décrit par une mesure sur l'espace de leurs vitesses. Précédemment on a justifié et résolu l'équation fondamentale que vérifie cette mesure. Les solutions sont étudiées au point de vue de leur évolution au cours du temps et de leur comportement asymptotique. *Résumé de l'auteur.*

Barker, J. A. The cell theory of liquids. III. Fusion. Proc. Roy. Soc. London. Ser. A. **240** (1957), 265–273.

An analysis of correlation effects in the cell model of liquids indicates the existence of a liquid-solid transition. Some numerical estimates of the transition temperature, using a 12–6 potential as a model for argon, give a value about 30% above the observed temperature. The author believes that the most likely cause of this discrepancy is the non-additivity of the interatomic potentials.

G. Newell (Providence, R.I.).

Peretti, Jean. Propriétés analytiques de la fonction $\mathcal{F}(z)$ attachée à un réseau cristallin à une dimension. C. R. Acad. Sci. Paris **244** (1957), 1311–1313.

In a previous note the author has introduced a certain analytic function simply related to the elastic frequency distribution of a crystal lattice [same C. R. **241** (1955), 461–463; MR **17**, 568]. This function is here shown to be algebraic and to have only branch points of order 2, which are simple poles in the square root of the displacement. *L. Van Hove* (Utrecht).

★ **Elliott, R. J.; and Marshall, W. The theory of critical scattering.** International conference on current problems in crystal physics. pp. 73–79. Massachusetts Institute of Technology, Cambridge, Mass., July 1–5, 1957.

The behaviour of a ferromagnetic system of spins is studied near the Curie point for the Ising and Heisenberg models, using the Bethe-Peierls method. The correlation between spins is calculated, also for the case that the two spins are taken at different times. This correlation determines the large excess in the magnetic scattering of neutrons near the Curie point. It is stressed that a similar sort of critical scattering will be found in alloys near an order-disorder transition. The paper is brief and few derivations are given. The methods used apply also to antiferromagnetic systems. *L. Van Hove* (Utrecht).

Farquhar, I. E.; and Landsberg, P. T. On the quantum-statistical ergodic and H-theorems. Proc. Roy. Soc. London. Ser. A. **239** (1957), 134–144.

Von Neumann [Z. Physik **57** (1929), 30–70] has established a quantummechanical ergodic theorem. He calculated the difference between the expectation value of a macroscopic observable and its microcanonical average, and showed that the time average of the square of this difference is less than a certain upper bound. The time averaging process involved two assumptions concerning the system, viz., that it is not degenerate and that the energy differences between pairs of levels are never equal. The above-mentioned upper bound is not necessarily small, but von Neumann introduced an averaging over observers and showed that it is small for the great majority of observers.

In the present paper it is shown that the two assumptions concerning the system are redundant. It is true that

when they are violated extra terms occur in the time averages, but these are subsequently annihilated by averaging over the observer. The only condition left is that the number of quantum states per phase cell is large, i.e., that the observables are truly macroscopic. {It seems that a stronger statement can be made: the process of averaging over observers makes even the time averaging unnecessary.} On the other hand, the existence of non-ergodic systems shows that the observer averaging cannot always be right. The conclusion is that averaging over observers is too drastic a process to find out which physical properties are responsible for ergodic behavior.

N. G. van Kampen (New York, N.Y.).

Stratonovich, R. L. On distributions in representation space. Soviet Physics. JETP **4** (1957), 891–898.

The concept of representation distributions is introduced for quantum systems. The quantum generalization is found for certain relations in the theory of correlated random points. The general formulas are illustrated in a concrete way for the cases of distribution of spin orientation. *Author's summary.*

Brueckner, K. A.; and Sawada, K. Bose-Einstein gas with repulsive interactions: General theory. Phys. Rev. (2) **106** (1957), 1117–1127.

An attempt is made to determine the low energy states of a Bose gas with repulsive interaction. The ordinary perturbation methods give rise to integrals that diverge for low wave numbers. Therefore the Hamiltonian is replaced with a simplified one, which would give rise to the same divergent terms, but which can be treated exactly. The excited states of this simplified Hamiltonian have an energy proportional to their momentum, as expected for phonons. For the energy perturbation of the ground state, one finds

$$\Delta E=(2\pi N\rho a/m)\{1+(128/15\pi^{\frac{1}{2}})(\rho a^3)^{\frac{1}{2}}+O(\rho a^3)^{\frac{1}{2}}\};$$

here a is the scattering length associated with the interaction force, and O represents terms not yet evaluated. The bulk of this correction is due to the interaction of each virtual excited pair of particles with the medium of unexcited particles.

This suggests that the case of strong interaction can also be treated by replacing the first-order propagator for excited particles with a modified propagator, which includes the interaction with unexcited particles. This method was developed for a Fermi gas by Brueckner and Levinson [Phys. Rev. (2) **97** (1955), 1344–1352]. Thus a formal expression for a "model Hamiltonian" can be given in terms of this modified propagator, describing the motion of dressed particles. It turns out that they again obey the energy-momentum relation of phonons. — The differences of this case those of from of Fermi statistics and of attractive forces are discussed. *N. G. van Kampen.*

Brueckner, K. A.; and Sawada, K. Bose-Einstein gas with repulsive interactions: Hard spheres at high density. Phys. Rev. (2) **106** (1957), 1128–1135.

The method of the paper reviewed above is applied to the case of hard spheres. The propagator is evaluated explicitly, although with a number of approximations. The low excited states have the character of phonons, which move unhampered through the unexcited medium, but are scattered by mutual interaction. The energy as a function of momentum shows a dip, as postulated by Landau to describe the effect of "rotons". The results are claimed to be reliable even for densities near to close packing [see, however, N. Hugenholtz, Physica **23** (1957), 533–545]. *N. G. van Kampen* (New York, N.Y.).

Lee, T. D.; Huang, Kerson; and Yang, C. N. Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties. Phys. Rev. (2) **106** (1957), 1135–1145.

The problem of a Bose gas of hard spheres is approached by an approximation method for low densities. The atoms, e.g. helium, are replaced by point particles without attraction, but with a repulsion represented by a "pseudo-potential" [see K. Huang and C. N. Yang, Phys. Rev. (2) **105** (1957), 767–775; MR **18**, 702]. The Hamiltonian of this model system is further simplified by discarding all off-diagonal elements that vanish if the total number of particles tends to zero. From the remaining Hamiltonian, the ground state and the lowest excited states can be calculated algebraically. The pair distribution function is found and it is shown that the low excited states may be described in terms of phonons. The approximations made assume that the radius of the hard sphere is small compared to the distance between atoms, i.e., $\rho a^3 \ll 1$.

N. G. van Kampen.

Oppenheim, Irwin; and Ross, John. Temperature dependence of distribution functions in quantum statistical mechanics. Phys. Rev. (2) **107** (1957), 28–32.

An integro-differential equation for the temperature dependence of the Wigner distribution function is obtained by taking a Fourier transform of the Bloch equation for the temperature dependence of the density matrix. The result approaches the classical relation as the Planck constant h is made to vanish, but for finite h, it contains several additional terms. The authors apply two perturbation treatments to their equation, one involving expansion in powers of h, the other, expansion in powers of a parameter of the order of magnitude of the potential energy. A final section discusses the use of transformation functions which convert the density matrix and Wigner function at one temperature to those at a different temperature. This procedure has the advantage of automatically maintaining the proper symmetrization required for Fermi-Dirac or Bose-Einstein statistics. *S. Prager.*

Ezawa, H.; Tomozawa, Y.; and Umezawa, H. Quantum statistics of fields and multiple production of mesons. Nuovo Cimento (10) **5** (1957), 810–841.

The authors consider the state of thermal equilibrium for the theory of quantised fields and attempt to relate the phenomenological energy-momentum tensor of a relativistic fluid (of mesons) to the S-matrix theory. Previously, L. Landau [Izv. Akad. Nauk SSSR. Ser. Fiz. **17** (1953), 51–64] had inferred the multiplicity of meson production in very high energy meson-nucleon collisions from relativistic fluid theory. The above paper considers the degree to which the phenomenological theory may be based on a perturbation theory of the statistics of quantum fields. The authors find that the equation of state for thermal equilibrium depends upon the meson-nucleon interaction assumed in the field theory. In particular, they show that at high temperatures the Stefan-Boltzmann law ($E=\sigma T^4$) holds only approximately for interactions which are renormalizable, and not at all for non-renormalizable interactions. Similarly, the relation $P=\frac{1}{3}E$ is shown not to hold in general. *D. Falkoff.*

Ecker, G. Das Mikrofeld in Gesamtheiten mit Coulombscher Wechselwirkung. Z. Physik **148** (1957), 593–606.

The statistical distribution of the local electric field in a plasma is investigated. It was calculated in a classical paper by Holtsmark [Ann. Physik. (4) **58** (1919), 577–630] under the assumption that the charge carriers are completely uncorrelated. The present paper introduces the correlations approximately by replacing the Coulomb field of each carrier by the screened field as originally calculated by Debye and Hückel [Phys. Z. **24** (1923), 185–206]. *L. Van Hove* (Utrecht).

Péneloux, André. Remarques sur l'hypothèse fondamentale de la théorie des processus irréversibles de K. Popoff. C. R. Acad. Sci. Paris **244** (1957), 2589–2591.

A criticism is presented of the endeavor of K. Popoff [Les bases mathématiques de la théorie des processus thermodynamiques irréversibles, Mémor. Sci. Phys., no. 63, Gauthier-Villars, Paris, 1956; MR **18**, 850] to provide a rationalization of irreversible thermodynamics based on Newtonian mechanics. It suffices to show incompatibility of the proposed theory with the special case of the behavior of a damped pendulum. *N. A. Hall.*

See also: Physical Applications: Finzi. **Elasticity, Plasticity:** Ziegler. **Fluid Mechanics, Acoustics:** Alpher. **Relativity:** Synge; Clemmow and Willson.

Elasticity, Visco-elasticity, Plasticity

★ **Babuška, Ivo; Rektorys, Karel; a Vyčichlo, František. Matematická theorie rovinné pružnosti. [Mathematical theory of plane elasticity.]** Nakladatelstvi Československé Akademie Věd, Praha, 1955. 527 pp. 39 Kčs.

This book presents the collective work of the authors based on the manuscript of the first author which was discussed in the Mathematical Institute of the Czech Academy of Science.

The subjects treated in this book are divided into five chapters. The initial chapter deals with the basic concepts (stress and strain, Hooke's law, St. Venant's conditions of compatibility). The second chapter is devoted to the stress function and to the plane problems of the theory of elasticity (Airy's function, biharmonic functions, the first and second problem of the theory of elasticity). The next chapter deals with the solved problems of plane elasticity (Mushelishvili's integral equations, Lauricelli-Sherman's equations, the approximate methods, St. Venant's principle). In the fourth chapter the method of conformal mapping is discussed (i.e. the transformation and approximate solution of the stress functions). The last chapter is a mathematical appendix on the theory of functions of complex variables, integral equations, etc.

There are a few practical examples. The discussion is mainly based on the Russian modern literature of this region. The book is particularly recommended to young Czech stress analysts and applied mathematicians.

D. Rašković (Belgrade).

★ **Kikukawa, Makoto. On plane-stress problems in domains of arbitrary profiles.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 5–9. Science Council of Japan, Tokyo, 1954.

★ **Kikukawa, Makoto. On plane-stress problems in domains of arbitrary profiles. II.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 149–152. Science Council of Japan, Tokyo, 1955.

★ **Negoro, Shosaburo; and Sekiya, Tsuyoshi. On a method of solving the so-called elastic plane stress problems.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 179–184. Science Council of Japan, Tokyo, 1952.

★ **Kikukawa, Makoto. On a method of calculating stress concentration in an infinite plate with a hole of an arbitrary profile.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 185–191. Science Council of Japan, Tokyo, 1952.

★ **Higuchi, Masakazu. Calculation of the stresses of the orthotropic strip with a hole.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 193–198. Science Council of Japan, Tokyo, 1952.

Litvinov, N. V. Investigation and application of the matrix of influence numbers of a biharmonic operator for solution of a problem on stresses in an infinite strip. Ukrain. Mat. Ž. **7** (1955), 388–402. (Russian)

A continuation of the article reviewed in MR **17**, 792. For the biharmonic problem on an infinite strip some properties of the influence numbers are presented. By a series of examples involving reduction of the graphs of the strains in various cross-sections, the practical use in engineering calculations of the influence numbers so obtained is demonstrated.

G. N. Savin (RŽ Mat 1957, no. 492).

Saito, Hideo. Stress in a plate containing infinite parallel rows of holes. Z. Angew. Math. Mech. **37** (1957), 111–115. (German, French and Russian summaries)

The paper deals with the stress distribution in an infinite plate under plane stress, which is perforated by an infinite number of circular holes with centers placed periodically on straight lines parallel to two directions x and y in the form of a rectangular grid. If the components of stress at an arbitrary point with the coordinate $\zeta=x+iy$, are denoted by $\sigma_x, \sigma_y, \tau_{xy}$, then, because of the periodicity of the plate geometry, the complex representation of the state of stress can be given in the following double series form

$$\sigma_x+\sigma_y=4\mathrm{Re}[\phi(\zeta)+\sum_m\sum_n \phi(\zeta-q_{mn})],$$

$$\sigma_y-\sigma_y+2i\tau_{xy}=2[\zeta\phi'(\zeta)+\psi(\zeta)+ +\sum_m\sum_n (\zeta-\bar{q}_{mn})\phi'(\zeta-q_{mn})+\sum_m\sum_n \psi(\zeta-q_{mn})],$$

where q_{mn} is the coordinate of the center of the hole in the mth row and the nth column, and where the infinite summation excludes $m=n=0$. $\phi(\zeta)$ and $\psi(\zeta)$ are two complex functions, holomorphic in the whole plane outside the hole at $\zeta=0$, which can therefore be developed in a Laurent series. The constants occurring in these series are to be determined by the boundary conditions along the edges of the circular holes. These determine an infinite system of linear equations. The author gives an approximate solution by developing the unknowns in a power series in terms of the radius of the holes. The question of the convergence of these expansions is only mentioned in the sense of calculating the first terms numerically.

W. Schumann (Zürich).

Winslow, A. M. Stress solutions for rectangular plates by conformal transformation. Quart. J. Mech. Appl. Math. **10** (1957), 160–168.

The method of two-dimensional stress analysis in terms of a complex variable given by N. I. Muskhelishvili [see, e.g., Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, part V; MR **11**, 626; **15**, 370] is applied to the problem of a rectangular plate stressed by arbitrary distributed forces along the four edges which act in the plane of the plate. Here the two analytic functions from which the components of stress can be derived, satisfy the boundary conditions of the first fundamental problem, in which a function appears which is approximately the integral of the edge tractions along the boundary. The problem is to extend this last function, after subtraction of a trivial polynomial part, conveniently into the interior of the domain under consideration. This is done by a conformal mapping of the rectangular area onto the unit circle and application of Cauchy's integrals. A development in power series, where the well-known Legendre's coefficients appear in the function defining the conformal transformation, leads directly to a solution for the stresses. The examples of a square and a rectangular plate with parabolic distributed tractions along two opposite edges are calculated numerically to show the procedure, and comparison is given with the results obtained by strain-energy methods [see, e.g., S. Timoshenko and J. N. Goodier, Theory of elasticity, 2d ed., McGraw-Hill, New York, 1951, pp. 167–170; MR **13**, 599]. Further, upper and lower bounds for the stresses are found for the approximation by finite series. The author finally gives an interesting discussion concerning the convergence and the amount of numerical work. *W. Schumann.*

Muki, Rokurō. A three-dimensional problem of a semi-infinite elastic solid under the compressive action of a rigid body. Proc. Fac. Engrg. Keio Univ. **8** (1955), 68–81.

The author considers the problem of determining the distribution of stress in a semi-infinite elastic solid under the compressive action of a rigid body of arbitrary (non-axially symmetric) shape. Assuming that the contact are a between the solid and the rigid body remains circular in shape and using the method of Hankel transforms combined with Fourier series previously developed [Proc. 5th Jap. Congress Appl. Mech., 1955, Science Council of Japan, Tokyo, 1956, pp. 119–124], the author derives a solution in the general case. This solution is then applied to a detailed discussion of the stress distribution in the solid when its surface is indented by a slightly inclined flat-ended cylinder. The results of detailed numerical calculations of the stress components (for a Poisson ratio of $\frac{1}{4}$) are presented in a series of diagrams.

I. N. Sneddon (Glasgow).

Pacelli, Mauro. Contatto con attrito tra due corpi elastici di forma qualunque: compressione e torsione. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 155–184 (1957).

Hertz treated the problem of two elastic bodies pressed together. Mindlin, Lubkin and Cattaneo have considered the effect of applying torsion to the bodies. Cattaneo made the hypotheses that there exists an inner region of contact where sticking occurs, an outer where sliding is permitted and that the boundaries of the two regions are concentric co-axial ellipses; he solved the problem approximately. Using the same hypotheses, the present author obtains an exact solution. *D. R. Bland* (Manchester).

Uflyand, Ya. S. Contact problem of elasticity theory. Prikl. Mat. Meh. **20** (1956), 578–587. (Russian)

A new method of solving the following mixed boundary-value problem for half-space ($z \geqq 0$) is shown: in the domain $z=0$, $r<a$ ($r=\sqrt{(x^2+y^2)}$), the displacements u, v, w are given; while in the domain $z=0$ ($r>a$) the stresses $\sigma_z, \tau_{zx}, \tau_{zy}$ are given.

The method consists of the application of toroidal coordinates and of the representation of Fourier-series coefficients in the polar angle φ by the integrals of Mehler [Math. Ann. **18** (1881), 161–194], and Fok [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **39** (1943), 253–256; MR **5**, 181]. The author claims that his method is essentially simpler than the other known solutions of the same problem.

As an application of the general results obtained he analyzes in full generality the following contact problem: on a rigid punch with plane circular base of radius a, in the presence of friction with the elastic half-space, a tangential force T (in the direction of the x-axis in the xy-plane) and a momentum M (in the xz-plane are exerted.)

T. P. Anđelić (Belgrade).

Blackburn, W. S.; and Green, A. E. Second-order torsion and bending of isotropic elastic cylinders. Proc. Roy. Soc. London. Ser. A. **240** (1957), 408–422.

Using a second order elasticity theory, the authors discuss torsion and bending of cylinders of very general shape, and, without solving the relevant equations, obtain formulas for the fractional elongation in torsion and change in length of the line of centroids in bending. In both cases, the problems to be solved are reduced to the solution of a linear elasticity problem together with the solution of a single boundary value problem involving two complex functions. *J. L. Ericksen.*

Schwarze, G. Allgemeine Stabilitätstheorie der Schalen. Ing.-Arch. **25** (1957), 278–291.

The paper deals with a general theory of elastic stability of shells, using tensor notation. After having developed the state of the deformed shell in the usual way, the author describes the corresponding state of equilibrium, which leads to general equations governing the problem. Then the stress-strain relations are discussed in detail. Finally the method of elastic energy is applied to the problem, thereby checking the earlier results and gaining the starting point for approximate solutions.

The paper gives all the tools, in a rather general way, for solving problems of elastic stability of shells but does not go into the calculations of special problems or give any numerical results. *W. Zerna* (Hanover).

Hodge, P. G., Jr.; and Papa, John. Rotating disks with no plane of symmetry. J. Franklin Inst. **263** (1957), 505–522.

This paper concerns the analysis of elastic stresses in a rotating disk, symmetric about the axis of rotation, but not about any plane normal to this axis. Bending moments are thus present as well as radial and tangential forces. The problem is reduced to that of solving two simultaneous linear second order differential equations in two stress functions. The solution can be found analytically in certain cases, and numerically in others by a generalization of a method of Manson [NACA Rep. no. **871** (1947)]. However, the authors also present an iterative method of solution, which is more efficient numerically, at least when the bending effects are not too large.

W. E. Boyce (Troy, N.Y.).

Windels, R. Biegetheorie der Rotationsschale mit flacher, kreisförmiger Erzeugender. Ing.-Arch. **25** (1957), 164–173.

Derivation of a system of differential equations of the linear theory of bending for orthotropic nearly cylindrical shells, for the case of shells of revolution with shallow circular meridian curves. The equations of the theory are reduced to two simultaneous fourth order equations, analogous to what is known for shallow shells. The paper concludes with a system of particular solutions which are suitable for the analysis of edge effects in the shells under consideration. *E. Reissner* (Cambridge, Mass.).

Roth, Werner. Der Membranspannungszustand in einer Rohrschlange. Z. Angew. Math. Mech. **37** (1957), 115–128. (English, French and Russian summaries)

The paper begins with a derivation of the scalar membrane equilibrium equations for a hollow spiral shell. Through proper choice of the variables the coordinate system on the shell surface becomes orthogonal. Explicit solutions are obtained for a shell subjected to (1) uniform internal pressure, and (2) forces and moments of such nature as to allow stresses non-varying in the direction of the helical centerline of the shell. It is found for the latter problem that, in general, the stresses which are obtained become infinite at certain points of the shell. (This means that for this problem membrane theory is effectively inapplicable and that wall bending action must be considered.) — Also considered is a method of solution of the equations of the paper in the event that stresses do vary in the direction of the centerline of the shell.

E. Reissner (Cambridge, Mass.).

Yuan, S. W.; and Ting, L. On radial deflections of a cylinder subjected to equal and opposite concentrated radial loads. Infinitely long cylinder and finite-length cylinder with simply supported ends. J. Appl. Mech. **24** (1957), 278–282.

The problem of the title is considered on the basis of two different sets of differential equations of the linear theory of thin, elastic, circular cylindrical shells. One set is due to L. H. Donnell [NACA Rep. no. **479** (1934)] and the other due to W. Flügge [Ing.-Arch. **3** (1932), 463–506].

Flügge's equations represent a refinement of the classical equations of A. E. H. Love while Donnell's equations represent a simplification of Love's equations which is expected to be rational in the event that all significant changes in stresses and displacements occur over axial and circumferential distances which are small compared with the radius of the middle surface of the shell. A solution of the problem mentioned in the title had previously been given, on the basis of Donnell's equations, by the first author [Quart. Appl. Math. **4** (1946), 13–26; MR **7**, 502]. In view of the fact that for this problem the validity of Donnell's equations is questionable, the present investigation is of interest. A numerical evaluation shows that the deflection of the load points according to the two theories is quite similar as long as the ratio of shell length to shell radius is sufficiently small but that the calculated deflections may differ as much as 25 percent for very long shells, for the case that the ratio of shell radius to shell thickness has the value 100. (It remains to obtain values for this percentage difference for the infinitely long shell, as a function of shell radius over shell thickness). *E. Reissner* (Cambridge, Mass.).

Bromberg, Eleazer. Non-linear bending of a circular plate under normal pressure. Comm. Pure Appl. Math. **9** (1956), 633–659.

Numerical solutions are obtained for a wide range of values of the normal pressure, and of the thickness of the plate by use of the differential equations of Föppl and von Kármán. Three types of boundary conditions are considered: 1. slope and circumferential strain zero, 2. bending moment and circumferential strain zero, and 3. radial stress and radial bending moment zero. In all three cases, edge effects occur when the plate is thin which cause difficulties in obtaining numerical solutions. An asymptotic treatment of such boundary layer effects, of the same general sort as was developed by Friedrichs and Stoker [Amer. J. Math. **63** (1941), 839–888; MR **3**, 223] leads to satisfactory numerical solutions; however, the author obtains the asymptotic solution in a form different from that of Friedrichs and Stoker in that he succeeds in combining the interior limit with the boundary layer solution [see also W. Z. Chien, Sci. Rep. Nat. Tsing Hua Univ. **5** (1948), 71–94; MR **10**, 218] to obtain a uniform approximation over the whole radius of the plate.

J. J. Stoker (White Plains, N.Y.).

Yanowitch, Michael. Non-linear buckling of circular elastic plates. Comm. Pure Appl. Math. **9** (1956), 661–672.

Two problems are treated, both in terms of the Föppl-von Kármán equations, in which the basic question is whether certain bent states of the circular plate in which the middle surface is a surface of revolution may become unstable at some value of a load parameter if the plate is permitted to have deflections which are not constant in the angular coordinate; i.e. if the constraint implied in assuming radial symmetry is relaxed. One of the problems is that treated by Friedrichs and Stoker [Amer. J. Math. **63** (1941), 839–888; MR **3**, 223] concerning the buckling, with radial symmetry, of the circular plate under radial pressure at its edge and subject to the boundary condition of vanishing deflection and bending moment. These authors found that a very thin plate buckles in such a way that there is a very high circumferential compression in a thin boundary layer at the edge of the plate, while the rest of the plate is nearly in a state of uniform tension; they were thus led to conjecture [J. Appl. Mech. **9** (1942), A-7-A-14; MR **3**, 288] that this state might in its turn buckle or wrinkle near the edge in modes having diametral nodes. This conjecture seems to be false: the author shows numerically that a second buckling is to be expected only if the condition of zero deflection at the boundary is given up and is replaced by the condition implied in supporting the plate by a distribution of lateral springs along its edge. The second problem treated is the stability of the circular plate bent into a surface of revolution under pressure normal to its face, with the boundary conditions of vanishing bending moment and radial stress (i.e. case 3 of the paper reviewed above). That such a state might become unstable under high normal pressure is made plausible by the fact that Bromberg (see the above review) shows that the circumferential membrane stress in this case is a compression over a ring at the edge of the plate which has a width of nearly one-third its radius; the author shows numerically that this state does indeed eventually become unstable when displacements not having radial symmetry are permitted. The method used by the author employs the second variation of the potential energy of the plate, and makes use of the ap-

proximate solutions of Bromberg and Friedrichs-Stoker which result from boundary layer theory.

J. J. Stoker (White Plains, N.Y.).

Sittig, E. Zur Systematik der elastischen Eigenschwingungen isotroper Kreiszylinder. Acustica **7** (1957),175–180.

Die elastische Grundgleichung des isotropen Festkörpers liefert für den Grenzfall des unendlich langen Kreiszylinders außer den einfachen Lösungen, welche die Typen der Torsions-, Longitudinal- und Biegeschwingungen beschreiben, auch noch kompliziertere. Es wird gezeigt, daß insgesamt eine dreifach abzählbare Schar von Resonanzen auftritt, sich also eine jede Resonanz durch ein Tripel ganzer Zahlen (n, p, q) indizieren läßt. Diese Indizes lassen sich mit den Schwingungsknoten auf der Zylinderoberfläche in Beziehung bringen und sind durch diese experimentell zu bestimmen.

Aus der Zusammenfassung des Autors.

Čelomeĭ, V. N. On the possibility of increasing stability of elastic systems with the aid of vibrations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 345–347. (Russian)

The author considers the interesting phenomenon of the transformation of a statically unstable system into a dynamically stable one by the addition of periodic forces of suitable frequency and amplitude. Oscillations are designated as stable if their amplitude remains bounded in time. The systems singled out for attention are those for which the equation of dynamic equilibrium is separable and for which the time-dependent portion of the displacement satisfies a given linear differential equation. Possible generalization of the analysis for the treatment of non-linear problems is indicated as well.

It is assumed that the frequency, ω, of the applied force is large compared with the fundamental frequency, Ω, of free vibrations of the system. A perturbation approach is used and a stability criterion derived which indicates that stability is possible for a steady load exceeding the static buckling load provided that a periodic force of suitable amplitude and frequency is superposed.

No particular physical system is discussed in detail. However, it is stated that the analysis would apply to the case of a thin bar under an axial load consisting of a steady part and a periodic part. The author is evidently unfamiliar with an earlier paper by S. Lubkin and J. J. Stoker [Quart. Appl. Math. **1** (1943), 215–236; MR **5**, 83] in which this problem is discussed with great care and in considerable detail for a bar with pinned ends. His results are less general than those of Stoker and Lubkin because of the restrictive assumption, $\Omega/\omega \ll 1$. A comparison of the two papers points up the fact that conclusions of a negative character which may be valid for the special case $\Omega/\omega \ll 1$, are not necessarily valid more generally.

For example, Čelomeĭ's stability criterion implies that instability will not occur in one of the higher vibrational modes unless the lowest mode is unstable as well. The work of Lubkin and Stoker shows that this is not always so and that a stable lowest mode does not necessarily mean a stable system.

Perhaps even more significant is a contradiction of another implication of Čelomeĭ's analysis. His stability criterion is such that a statically stable system cannot be made dynamically unstable by the addition of a periodic force. Lubkin and Stoker show that for a pin-ended bar, the addition of a suitable periodic force of small amplitude can produce unbounded oscillations even when the steady portion of the axial load is smaller than the Euler buckling load, and even, in fact, when the steady axial load is tensile. *P. Mann-Nachbar* (Palo Alto, Calif.).

Klein, Bertram. Shear buckling of simply supported plates tapered in planform. J. Franklin Inst. **264** (1957), 43–47.

A simple, approximating set of equations is derived for calculating the shear buckling loads of isosceles trapezoidal plates. A numerical example is worked for a typical plate. *From the author's summary.*

Shuleshko, P. A reduction method for buckling problems of orthotropic plates. Aero. Quart. **8** (1957), 145–156.

See the article in Austral. J. Appl. Sci. **8** (1957), 7–26 [MR **19**, 41] by the same author. The content of the two papers is essentially the same. *R. C. T. Smith.*

Bijlaard, P. P. Buckling under external pressure of cylindrical shells evenly stiffened by rings only. J. Aero. Sci. **24** (1957), 437–447, 455.

This paper presents calculations for determining the critical buckling pressure of ring stiffened cylindrical shells subjected to external pressure. Panel buckling strength (buckling of shell portion between elastically stable rings) is calculated by determining the equivalent length of the simply supported shell between stiffeners, accounting for the rotational restraints of the rings. As the author points out, this method yields adequate accuracy for shells where the ratio of the half wave lengths of the longitudinal and circumferential buckles is greater than 4 and therefore applies to shells of moderate length/radius ($l/a > 1$). In the case of very short shells ($l/a = .1$) the limited applicability of small deflection theory is accounted for by partially neglecting the curvature. For the case of general instability where both shell and rings buckle, the author uses the method of split rigidities in which the critical buckling pressures corresponding to two distinct modes of buckling are superposed. Rotational spring constants are calculated for rings having plate and box sections.

Included is a discussion of reduction factors for correcting the concentric elastic buckling stress for plasticity and out-of-roundness effects.

Close agreement between the several computations presented and test results indicate that this paper may be quite useful to the stress analyst. *A. P. Coppa.*

Stoneley, Robert. The transmission of Rayleigh waves across an ocean floor with two surface layers. I. Theor etical. Bull. Seismol. Soc. America **47** (1957), 7–12.

From the author's abstract: The theoretical part of this paper is a discussion of the propagation of waves of Rayleigh type in an elastic medium with a horizontal double surface layer, above which is a uniform layer of liquid. This model is based on seismic determinations of the velocities of explosion waves in the layers below the ocean bottom, and the equation giving the wave velocity as a function of wave length is derived as a determinantal equation of the eleventh order. *J. J. Stoker.*

Oldroyd, J. G. The rheology of some two-dimensional disperse systems. Proc. Cambridge Philos. Soc. **53** (1957), 514–524.

The problem treated by the author is that of relating the rheological properties of a composite material to those of its constituents when the latter consists of small, nearly

circular discs of one material widely dispersed in a thin sheet of another. He assumes that these materials are adequately described by linear viscoelasticity theory and takes into account surface tension in the sheet and at the interfaces separating the two media. *J. L. Ericksen.*

Ziegler, H. Thermodynamik und rheologische Probleme. Ing.-Arch. **25** (1957), 58–70.

The thermodynamic treatment of visco-elasticity given by the author is based on the assumption that the deformation can be decomposed into an elastic and an inelastic part. He investigates, in particular, the constitutive equations for viscous fluids which relate the components of the stress to the components of the rate of deformation. In the linear case the coefficients in these equations have certain symmetry properties. The author claims that this symmetry can be derived in general only by applying Onsager's reciprocity principle. In the analogous case of linearly elastic bodies the corresponding symmetry is a consequence of the existence of a strain energy function. Extending this analogy the author proposes to generalize Onsager's principle to the non-linear case by postulating the existence of a "velocity potential" playing a role analogous to the strain energy. {Reviewer's remark: The author does not include any kinematical considerations. He thus fails to realize that the special form of the constitutive equation for linear viscous fluids and hence the symmetry property follows from kinematical invariance requirements [cf. Noll, J. Rational Mech. Anal. **4** (1955), 3–81; MR **16**, 764]. *W. Noll* (Pittsburgh, Pa.).

Johnson, W. The plane strain extrusion of short slugs. J. Mech. Phys. Solids **5** (1957), 202–214.

The solution for a long slug of material, extruded through frictionless dies has been known for some time [R. Hill and S. J. Tupper, J. Iron Steel Inst. **159** (1948), 353–359]. The solution is a "steady state" one in which the plastic configuration relative to the die is independent of time. For very short slugs of material, this steady state phenomenon is no longer true.

The present paper gives a solution for this case, including the effects of wall friction. Solutions are presented in the form of slip-line fields, pressure curves, and deformed grid patterns. Photographs of experimental results are also exhibited, and qualitative agreement is observed between theory and experiment. In particular, the theory explains the phenomenon of piping or coring which is known to occur near the end of extruded slugs.
P. G. Hodge (Chicago, Ill.).

Johnson, W. Partial sideways extrusion from a smooth container. J. Mech. Phys. Solids **5** (1957), 193–201.

Solutions are given for various extrusion problems in which the die orifice is inclined at various angles to the direction of motion of the billet. The orifice may be at the center or at the edge of the billet. Slip-line fields are presented, and various quantities of interest are graphed as functions of the reduction ratio.
P. G. Hodge (Chicago, Ill.).

Bijlaard, P. P. Plastic buckling of simply supported plates subjected to combined shear and bending or eccentric compression in their plane. J. Aero. Sci. **24** (1957), 291–303.

Buckling formulas are derived using the following assumptions: deformation theory is applicable; the stress-strain diagram is similar to that for mild steel; yielding is determined by shearing energy; the nodal buckling lines are straight (enabling the use of oblique coordinates); the mode of buckling is sinusoidal longitudinally and can be represented by a combination of three sine terms in the oblique transverse direction. Energy methods are used to determine the buckling mode and hence the plastic reduction factors. *E. H. Mansfield* (Farnborough).

Grandori, Giuseppe. Formulazione variazionale della condizione di plasticità per continui idealmente elasto-plastici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 541–546.

In the reviewer's opinion this paper is no more than a restatement of the well-known extremum principle concerning kinetically admissible strain rate fields.
D. R. Bland (Manchester).

Marciniak, Zdzisław. Graphical representation of states of stress and strain. Arch. Mech. Stos. **9** (1957), 261–274. (Polish and Russian summaries)

The representation is on the deviatoric plane. It can be used to solve very simple elasto-plastic problems.
D. R. Bland (Manchester).

Heldenfels, Richard R.; and Vosteen, Louis F. Approximate analysis of effects of large deflections and initial twist on torsional stiffness of a cantilever plate subjected to thermal stresses. NACA Tech. Note no. 4067 (1957), 36 pp.

The authors deal with a rectangular cantilever plate acted upon by thermal stresses and an applied twisting moment. The von Karman large deflection equations are used, and the plate is assumed to be initially twisted. The analysis is essentially of the Rayleigh type: a variational principle is used in conjunction with assumed functions for the stresses, displacement, and temperature distribution. These functions are chosen in part to conform with experimental conditions, and in part on the basis of a previous small-deflection study. There results an approximate relation involving the twist, initial twist, frequency of torsional vibration, temperature difference between free edges and center line, and applied moment. This equation is then studied in several cases of physical interest, and the results presented in graphical form; good agreement with experimental data is indicated. To the reviewer, the most interesting conclusions were those dealing with the effect of initial twist on the frequency of free vibrations. The thermal stresses act directly to reduce the frequency, but also tend to increase the initial twist which in turn serves to increase the frequency. The authors present curves illuminating the interaction of these two effects.
W. E. Boyce (Troy, N.Y.).

Lessen, M. The motion of a thermoelastic solid. Quart. Appl. Math. **15** (1957), 105–108.

Thermodynamic considerations are used to derive the equations of thermo-elasticity assuming infinitesimal displacements. The propagation of coupled displacement and temperature waves is investigated. *E. H. Mansfield.*

Weiner, J. H. A uniqueness theorem for the coupled thermoelastic problem. Quart. Appl. Math. **15** (1957), 102–105.

A uniqueness theorem is presented for the coupled thermo-elastic problem formulated by W. Voigt [Lehrbuch der Kristallphysik, Teubner, Leipzig-Berlin, 1910]. The author states that his theorem may be readily generalised to include the most general linear thermal and mechanical boundary conditions. *E. H. Mansfield.*

Nowacki, Witold. Thermal stresses in cylindrical shells. I. Arch. Mech. Stos. **8** (1956), 69–83. (Polish. Russian and English summaries)

Thrun, Zygmunt. Thermal stresses and strains in plates resting on elastic foundation. Rozprawy Inż. **4** (1956), 71–86. (Polish. Russian and English summaries)

Thrun, Zygmunt. Plates subjected to the action of temperature with horizontal reaction of the elastic foundation. Rozprawy Inż. **4** (1956), 87–97. (Polish. Russian and English summaries)

See also: Partial Differential Equations: Babakova; Kozyrev; Eringen. **Differential Geometry:** Gallissot. **Computing Machines:** Lazaryan. **Physical Applications:** Finzi.

Structure of Matter

Bloom, Stanley. Effects of radiation damping on spin dynamics. J. Appl. Phys. **28** (1957), 800–805.

The paper is an improvement over a similar treatment of the same problem by Bloembergen and Pound [Phys. Rev. (2) **95** (1954), 8–12]. In the present paper the radiation damping field and the applied driving field are not lumped together, but instead the spin system is regarded as such and its Bloch equations are augmented with terms due to the radiation damping field. The equations are solved for four cases of usual interest. It is shown that for steady-state slow passage the radiation damping lowers and broadens the absorption and dispersion signals. For adiabatic fast passage, the effects of the damping are a dip in the dispersion signal at the line center, and the creation of a finite absorption signal. It is also shown that if the radiation damping field exceeds the rf driving field, a complete adiabatic inversion is no longer possible. A discussion is given of the conditions under which delayed power peaks occur in the free-precession signal. Finally the case of the driven nonsteady state is taken up, and it is shown that if the radiation damping is large and the system is driven from an initially "negative temperature" condition, then the stimulated emission may become quite large. *M. J. Moravcsik* (Upton, N.Y.).

Mitra, G. B.; and Gokhale, B. V. Elastic and force constants of a simple tetragonal crystal. J. Chem. Phys. **26** (1957), 1655–1656.

The case of a single atom in a tetragonal unit cell is considered. Two sets of equations of motion of an atom executing lattice vibrations have been set up. Each set contains three equations describing the motion in the principal crystallographic directions. In both sets, the forces are expressed as linear combinations of the second derivatives of the displacements of the atom with respect to its position. In the first set the coefficients are linear combinations of the elastic constants of the crystal. In the second set the coefficients are linear combinations of the force constants. On equating the coefficients of the two sets of equations, the force constants are expressed in terms of the elastic constants.

Interactions of the atom with its first four nearest neighbors are considered, giving rise to four force constants. By considering free electrons to take part in the propagation of the elastic waves, a fifth force constant is obtained. *W. M. Macintyre* (Aberdeen).

Dresselhaus, G. Optical absorption band edge in anisotropic crystals. Phys. Rev. (2) **105** (1957), 135–138.

In studying the electronic properties and, especially the transport phenomena, of semi-conductors and crystals, one must have a knowledge of the structure of the energy bands (levels) of these materials. From the optical properties one can, however, provide useful information about the energy band structure. The purpose of this paper is to study the shape of the optical absorption band edge of semiconductors and hexagonal crystals resulting from direct and indirect transitions. It is found that the absorption coefficient has the form $k \simeq (\hbar\omega - E)$, where ω is the angular frequency of the external radiation, and E is related to the minimum band gap energy of the crystal. For direct electronic transition, ($k=0$) $n=1/2$ and for forbidden transition $n=3/2$. The indirect transitions can be accounted for by $n=2$. In deriving such a relation for the absorption coefficient for hexagonal crystals of a certain symmetry type, the author proceeds by expanding the perturbed wave function near $k=0$ (k is the propagation vector) and then calculates the matrix elements for dipole transition. The absorption coefficient is found to be different for polarizations parallel and perpendicular to the hexagonal axis. In the former case, $n=1/2$, and in the latter $n=3/2$, provided $\hbar\omega - E_g < E$ (E is the energy interval where the two absorptions coincide). Finally, the author discusses the experimental results obtained for CdS and Te crystals on the basis of the preceding analysis.

N. Chako (Flushing, N.Y.).

Hughes, Edward W. A new type of inequality relationship between unitary structure factors. Acta Cryst. **10** (1957), 376–377.

By using a Fourier series expansion for the function $|\cos 2\pi x|$, the following expression is obtained:

$$|U_H| \leq (2/\pi)\{1 - 2\sum_{n=0}^{\infty}[(-1)^n/(4n^2-1)]U_{2nH}\},$$

where U_H is the unitary structure factor for the plane hkl. Between $0.28 \leq U_H \leq 0.9$ this relationship is more likely to fix the sign of U_{2H} than the Harker-Kasper inequality, $2U_H^2 - 1 \leq U_{2H}$. Below this range the Harker-Kasper inequality will be more efficient, but the efficiency with which signs can be fixed in that region is low in any case. Above this range, the Harker-Kasper inequality [Acta Cryst. **1** (1948), 70–75; MR **12**, 496] will, in general, be more efficient.

By also using the Fourier series expansion for $|\sin 2\pi x|$, another inequality is obtained:

$$|U_H \pm U_K| \leq (4/\pi^2)\sum_{n,m=0}^{\infty}\sum[(\mp 1)^{n+m}/(4n^2-1)(4m^2-1)] \times [U_{(n+m)H+(n-m)K} + U_{(n-m)H+(n+m)K}].$$

During the derivation the assumption was made that the fraction of the scattering power on any given atom was a positive constant. *W. Macintyre* (Boulder, Colo.).

Mattis, Daniel Charles. Phonon free path in an isotopic mixture. Phys. Rev. (2) **106** (1957), 721–722.

In einem regulären Kristall, der aus einer einzigen chemischen Atomsorte aufgebaut ist, jedoch als Verunreinigung isotope Atome eines anderen Atomgewichtes, regellos verteilt, enthält und der demzufolge ebene Wellen streut, wird die freie Weglänge l eines Phonons nach der Newtonschen Mechanik und auch nach der Quantenmechanik berechnet. Es ist zu erwarten, dass die Resultate der ersteren Theorie bei hohen und die der letzteren bei tiefen

Temperaturen gültig sein werden. Aus der klassischen Berechnung, die aus den bekannten Schwingungsgleichungen ausgeht, folgt $l=a/(3.3\varepsilon)$, wo a die Gitterkonstante bedeutet und $\varepsilon=\langle(m_l-m)/m\rangle_{rms}$ ist. m bedeutet das mittlere Atomgewicht im Kristall, m_l die von einem herausgegriffenen Atom und der Bruch ist über alle Atome des Kristalls zu mitteln. Aus der Quantenmechanik folgt dagegen $l=a/3\pi\varepsilon$, also fast das selbe Resultat. Der Verfasser bemerkt jedoch in einer Fussnote, dass diese Berechnung noch ergänzungsbedürftig ist.

T. Neugebauer (Budapest).

Nijboer, B. R. A.; and DeWette, F. W. On the calculation of lattice sums. Physica **23** (1957), 309–321.

A short and straightforward method for the conversion of slowly converging lattice sums into expressions with good convergence is presented. As an illustration, a well-known expression for the Madelung constant is rederived. The method is then applied to two general types of lattice sums, the latter of which has not been treated before.

W. Nowacki (Bern).

Schafroth, M. R.; Butler, S. T.; and Blatt, J. M. Quasi-chemical equilibrium approach to superconductivity. Helv. Phys. Acta **30** (1957), 93–134.

The problem of superconductivity in metals has been attacked in the past by means of a Bose-gas model, by assuming resonance in the scattering of electrons in the metal [Schafroth, Phys. Rev. (2) **96** (1954), 1149, 1442; **100** (1955), 463–475; V. L. Ginzburg, Uspehi Fiz. Nauk **48** (1952), 25–118]. In the present paper the authors propose a new systematic approach to this phenomenon. This is essentially a quantummechanical generalization of the Ursell method [Proc. Cambridge Philos. Soc. **23** (1927), 685–697]. They propose a system of interacting Fermi particles; the correlations between the particles are produced by the interaction (so-called dynamical correlations) and by the statistics of the particles (statistical correlations). The writers neglect all dynamical correlations between more than pairs of particles and include all statistical correlations. They show that this approximation is valid in two cases: (a) in a dilute gas of atoms capable of forming diatomic molecules we obtain the thermodynamical theory of chemical equilibrium; (b) the electron gas in metals furnishes an extension of the free-electron theory. The success of the approximation in this case depends on the quenching effect of the Pauli principle, according to which the scattering process between electrons does not occur. Some remarks on the subject of the applicability of this quasichemical approximative method to the two cases mentioned above seem to justify the proposition. The Appendix contains a resume of thermodynamical theory of chemical equilibrium in perfect gases.

M. Z. v. Krzywoblocki (Urbana, Ill.).

See also: Statistical Thermodynamics and Mechanics: Peretti; Elliott and Marshall. **Fluid Mechanics, Acoustics:** Chandrasekhar and Donnelly; Chandrasekhar. **Optics, Electromagnetic Theory, Circuits:** Wild.

Fluid Mechanics, Acoustics

Pignedoli, Antonio. Sui fondamenti della teoria dei vortici e sui vortici cilindrici in particolare. Confer. Sem. Mat. Univ. Bari **20** (1956), 30 pp.

An expository lecture. *L. M. Milne-Thomson.*

Crabtree, L. F. Effects of leading-edge separation on thin wings in two-dimensional incompressible flow. J. Aero. Sci. **24** (1957), 597–604.

A summary about flows over a two-dimensional airfoil with a laminar bubble attached to it near the leading edge. *I. I. Kolodner* (Albuquerque, N.M.).

Couchet, Gérard. Mouvement d'un profil en présence d'une paroi. Rech Aéro. no. **59** (1957), 13–19.

The author considers the following two-dimensional flow topics in his study: The potential for a general profile in the presence of a wall of arbitrary shape; the potential for a general profile in the presence of a straight wall; the aerodynamics for the straight wall obstacle; the uniform motion of the profile parallel to the wall; the plate profile computed and comparison of results to those of Tomotika, Nagamiya, etc. A set of integral equations is developed for the general case. For the straight wall, successive approximations with neglect of higher (above 3rd order) terms reduce the above set to differential equations from which the potential is found in a Laurent expansion. This is followed by computation of formulas for the resultant force and pitching moment in terms of the usual quantities ρ, ϕ, τ etc. The results are further reduced for the straight line segment airfoil. Formulas are developed for the force and pitching moment on a general airfoil which moves parallel to a straight wall and on which the Kutta-Joukowsky condition is usable. When this profile is a line segment, the computed force agrees exactly with the results of Tomotika and Nagamiya. *M. G. Scherberg.*

★ **Birkhoff, Garrett; and Zarantonello, E. H. Jets, wakes, and cavities.** Academic Press Inc., Publishers, New York, 1957. xii+353 pp. $10.00.

This monograph is the first comprehensive review of a field that has been actively studied by hydrodynamicists and mathematicians for almost a century. It is an attempt to present systematically all sides of the subject and is sufficiently complete, especially with its exhaustive bibliography, to serve as a valuable reference work. However, it is apparent that the authors face a serious expository problem in the unevenness of the material — the level of the theory varies from the exact and mathematically mature (for cavitational flow) to the empirical (for wakes in real fluids) — and as a result they are led to remark, "the presentation retains much of the heterogeneous character of its original sources".

The volume seems to fall naturally into several parts which are distinguished from one another both in subject matter and in the type of theory involved. Chapters 2, 3, and 5 are concerned with exact solutions of plane free boundary problems obtained by the hodograph method. A more complete account is hard to imagine. An innovation here is the systematic study of classes of free surface flows according to their hodographs. Chapters 4, 6, and 7, which will be of most interest to mathematicians, present the very attractive qualitative theory of jet and cavity phenomena and several proofs of existence and uniqueness. The authors contribute, among other things, a new and relatively simple existence proof (based on Leray-Schauder) with a unified approach to several cavity models. The variational method is given in detail except for the essential proof of analyticity of the free streamline. A chapter on effective computation, based largely on the authors' own important work in this area, provides a useful collection of numerical results on plane free boundary problems. Chapters 8, 10, and 11 are devoted to the

comparatively limited theoretical results on compressibility and gravity, and on axially symmetric and unsteady flows. The final chapters (12–15) discuss various developments — mostly empirical and semi-empirical — concerning the theory of wakes and related topics.

D. Gilbarg (Stanford, Calif.).

Davies, H. J.; and Ross, A. J. A jet deflected from the lower surface of an aerofoil. Quart. J. Mech. Appl. Math. **10** (1957), 291–301.

Es wird die zweidimensionale, inkompressible, reibungslose Strömung um eine in Längsrichung angeströmte ebene Platte ermittelt, an deren Unterseite ein Strahl unter einem vorgegebenen Winkel gegen die Plattenrichtung austritt. Für die Rechnung werden die Platte und der Strahl durch eine Wirbelverteilung ersetzt. Aus der Bedingung, dass die Normalkomponenten der von den Wirbeln induzierten Geschwindigkeiten an der Platte verschwinden müssen, folgt eine Integralgleichung für die Wirbelverteilung und die Form des Strahles. Eine Umkehrung dieser Integralgleichung ergibt die explizite Darstellung der Wirbelverteilung an der Platte. Die Form des Strahles lässt sich dagegen nur iterativ ermitteln. Aus der Wirbelverteilung werden Auftrieb und Moment sowie die Druckverteilung an der Platte ermittelt. Ein Vergleich dieser Rechnung mit der Messung an einem dünnen, elliptischen Profil ergibt gute Übereinstimmung.

L. Speidel (Mülheim/Ruhr).

Kreiss, H. Some remarks about nonlinear oscillations in tidal channels. Tellus **9** (1957), 53–68.

The author's abstract is: In tidal rivers one observes an unsymmetry between flood and ebb current. This phenomenon is examined for channels with constant mean depth and mean cross-section area. The hydrodynamic equations are integrated over the vertical, and it is shown that one is not allowed to introduce the usual simplifications (linearization, neglection of friction) if one wants to describe this phenomenon. With the aid of a perturbation method the equations are integrated and it is shown that for, sufficiently small maximum ratio of the flow/mean depth one can approximate the solution by two harmonics. This approximation is examined in detail.

J. J. Stoker (White Plains, N.Y.).

Proudman, J. Oscillations of tide and surge in an estuary of finite length. J. Fluid Mech. **2** (1957), 371–382.

The author investigates the effects on a narrow basin of uniform cross section open to the sea at one end and closed at the other when a prescribed incident long wave enters the estuary from the sea. The equation of continuity is

$$\frac{\partial}{\partial x}((h+\zeta)u)+\frac{\partial \zeta}{\partial t}=0$$

and the equation of motion is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \zeta}{\partial x}=-\frac{k}{h}|u|u$$

where k is a numerical coefficient of friction and the other symbols have their usual meaning. Solutions of these equations for ζ and u are developed as perturbation series. The first order linear terms give the reflection of the wave at the head of the estuary. The second order terms are made determinate by assuming that the motion at the mouth reduces to a progression towards the sea. It is shown that when the first order elevation increases steadily to a maximum at the head of the estuary, the effect of the 'shallow water terms' is to make high water higher and earlier while the effect of the frictional term is just the opposite. Special emphasis is given to the effects on a short estuary.

H. Greenspan.

Hunt, J. N. A propos de l'onde solitaire d'amplitude finie. On the solitary wave of finite amplitude. Houille Blanche **10** (1955), 197–203.

The author writes the exact boundary conditions to be satisfied by a solitary wave in Levi-Civita's form and solves the resulting differential equations by a method of successive approximations. If c is the velocity, h the depth at infinity and a the amplitude of the wave, he finds

$$c^2=gh\left\{1+\frac{a}{h}-\frac{1}{20}\left(\frac{a}{h}\right)^2-\frac{3}{70}\left(\frac{a}{h}\right)^3\right\}.$$

The equation for the wave profile is given as a series in a/h up to $(a/h)^3$.

J. V. Wehausen (Berkeley, Calif.).

Yamada, Hikoji. Highest waves of permanent type on the surface of deep water. Rep. Res. Inst. Appl. Mech. **5** (1957), 37–52.

Levi-Civita's formulation of the problem of determining periodic progressing gravity waves of permanent type in water having a free surface is used as a basis for the numerical treatment of such waves when they have angular crests. The singularity at the crests, of the type given already by Stokes, is taken care of by subtracting an appropriate analytic function, after which the regular part of the solution is written formally as a power series (following transformations which map the flow on the unit circle minus a radius). The coefficients of 12 terms in the series are computed approximately by making use of the free surface conditions. Two different iteration schemes are used, with results which appear to be quite accurate.

J. J. Stoker (White Plains, N.Y.).

Yamada, Hikoji. On the highest solitary wave. Rep. Res. Inst. Appl. Mech. **5** (1957), 53–67.

The problem is the same as in the paper above, except that a solitary wave with a sharp crest is treated. The method is also the same, and the numerical results again seem to be accurate.

J. J. Stoker.

Littman, Walter. On the existence of periodic waves near critical speed. Comm. Pure Appl. Math. **10** (1957), 241–269.

The waves under consideration are symmetric long plane gravity waves of finite amplitude, known as cnoidal waves [Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, § 253]. By changing variables, the problem is reduced to a boundary value problem for the Laplace equation in an infinite horizontal strip, with boundary conditions which are non-linear on the top and linear on the bottom. This problem is reduced to solving a non-linear integral equation. The essential step of the existence proof consists in stretching the independent variable (and in corresponding changes in the dependent variables) by a small factor λ proportional to depth/wave length. One obtains a one-parameter family of integral equations depending on λ. For $\lambda=0$ the equation becomes simple and is transformed into an ordinary nonlinear differential equation which has a one-parameter set of solutions in terms of elliptic functions (the variable parameter being the modulus of the elliptic functions). The non-linear integral

equation is then solved by a perturbation procedure about $\lambda=0$. The number of free parameters is shown to be 2; asymptotic formulas for various quantities are derived, and the region of validity of the existence proof is discussed. The method is an extension of that used by Friedrichs and Hyers [Comm. Pure Appl. Math. **7** (1954), 517–550; MR **16**, 413] to prove the existence of the solitary wave. *F. Ursell* (Cambridge, Mass.).

Phillips, O. M. On the generation of waves by turbulent wind. J. Fluid Mech. **2** (1957), 417–445.

A theory is presented for the generation of waves upon a water surface, originally at rest, by a random pressure distribution of normal pressure associated with the onset of a turbulent wind. The pressure distribution is assumed to be homogeneous in the horizontal direction and (after the initial instant) stationary in time. Correlations between the air and water motions are neglected and the water is assumed to be inviscid, so that the motion of the water, starting from rest, is irrotational. The equations throughout this paper are linearized. It is found that waves develop most rapidly by means of a resonance mechanism which occurs when a component of the surface pressure distribution moves at the same speed as the free surface wave with the same wavenumber.

With each horizontal vector wave number $\mathbf{K}$ the pressure spectrum associates a 'convection velocity' $U_c(\mathbf{K})$ and a development time $\theta(\mathbf{K})$. The development of the waves is conveniently considered in two stages, $t \ll \theta(\mathbf{K})$ and $t \ll \theta(\mathbf{K})$. The expression given for the wave spectrum in the initial stage shows that the most prominent waves are ripples of wave lengths $\lambda_{cr}=1.7$ cm, corresponding to the minimum phase velocity c determined by gravity and surface tension, and moving in direction $\cos^{-1}(c/U_{cr})$ to that of the mean wind, where U_{cr} is the convection velocity corresponding to λ_{cr}. Observations qualitatively supporting this conclusion are quoted.

Most of the growth of gravity waves occurs in the second, or principal, stage of development which continues until the waves grow so high that non-linear effects become important. An expression for the wave spectrum is derived, and the theoretical mean-square amplitude is shown to be consistent with published oceanographic measurements. The difficult assumptions about the turbulent air flow are clearly and carefully presented.

A similar semi-empirical model had been studied earlier by C. Eckart [J. Appl. Phys. **24** (1953), 1485–1494; MR **15**, 662] who adopted a different statistical model (in which resonance is ineffective) for the turbulent air flow. He assumed a finite generating area and statistical stationarity in time. Here an equilibrium wave height is ultimately attained which, however, is much less than observed wave heights. From this it had to be concluded that normal pressures are comparatively ineffective and that most of the energy is put into the waves by mechanisms involving correlation (sheltering) or shear stresses for which a quantitative treatment is difficult. Phillips's work is a significant contribution since it shows that in the initial stages the normal pressures are important, possibly dominant. *F. Ursell* (Cambridge, Mass.).

Krzywicki, A. Sur un problème des forces et des moments exercés sur un obstacle par un fluide visqueux compressible. Studia Math. **16** (1957), 48–55.

In previous papers [Studia Math. **15**, 113–122 (1955), 174–181, 252–266 (1956); MR **18**, 352, 353] the author discussed the forces exerted by a compressible viscous fluid on a rigid body moving through it with a uniform translational motion. In the present paper the formulae are extended to cover any translational motion of the body, or any rotation. The main hypothesis with regard to the velocity of the fluid at infinity is the same as previously, viz. the kinetic energy of the fluid is finite. The paper has two parts, dealing respectively with plane motion and motion in space. *J. L. Synge* (Dublin).

Bhatnagar, P. L.; and Verma, P. D. On superposable flows. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 281–292.

If $\mathbf{q}$, p, Ω denote velocity, pressure and body force potential in incompressible viscous flow, the flows $(\mathbf{q}_1, p_1, \Omega_1)$ and $(\mathbf{q}_2, p_2, \Omega_2)$ are said to be superposable if a pressure distribution π exists such that the flow $(\mathbf{q}_1+\mathbf{q}_2, p_1+p_2+\pi, \Omega_1+\Omega_2)$ is a solution of the Navier-Stokes equations of motion, with the necessary modification in the boundary and initial conditions. If $\boldsymbol{\omega}$ denotes vorticity, the necessary and sufficient condition for superposability is found to be $\operatorname{curl}(\boldsymbol{\omega}_2\times\mathbf{q}_1)+\operatorname{curl}(\boldsymbol{\omega}_1\times\mathbf{q}_2)=0$. The author then proves that two axisymmetric flows will be superposable in three cases only (i) they are irrotational (ii) the vorticity of each flow is proportional to distance ϖ from the axis (iii) ω_1/ϖ is constant along the streamlines of flow 2 and ω_2/ϖ is constant along the streamlines of flow 1. Proceeding to two-dimensions, the flows superposable on a vortex, vortex doublet, spiral vortex and radial flow are obtained. *L. M. Milne-Thomson.*

Hughes, William F.; and Osterle, J. Fletcher. On the adiabatic Couette flow of a compressible fluid. Z. Angew. Math. Phys. **8** (1957), 89–96.

Let x, z be rectangular coordinates. Consider the flow of a viscous fluid in a narrow channel of length L parallel to the x-axis and of width h bounded by walls in relative motion at constant speed. The authors seek solutions of the lubrication equations with coefficient of viscosity that depends linearly on the temperature T. The fluid velocity is of the form $u=u(x, z)$, while the other flow functions are functions of x only. By assuming $T(x)$ is linear in x the authors are able to find an approximate integral of the differential equation for the pressure $P(x)$.
J. H. Giese (Aberdeen, Md.).

Power, G.; and Scott-Hutton, D. L. Slow shearing motion over a hollow. J. Franklin Inst. **263** (1957), 431–439.

The slow steady two-dimensional motion of a viscous incompressible liquid over a hollow in an infinite plane is considered, the motion being one of uniform shear apart from the disturbance caused by the hollow. The x, y-area is represented conformally on the circle $(u+iv)\leq 1$ by a relation of the form $x+iy=f(u+iv)$, where f is a rational function; as in similar examples [e.g., N. I. Muskhelisvili, Z. Angew. Math. Mech. **13** (1933), 264–282] a simple expression is found for the stream-function. A partial allowance is made for inertia by finding a second approximation to the stream-function from the exact equations of motion. *W. R. Dean* (London).

Potter, O. E. Laminar boundary layers at the interface of co-current parallel streams. Quart. J. Mech. Appl. Math. **10** (1957), 302–311.

The incompressible laminar boundary layer at the interface of two co-current parallel streams of different velocities is computed approximately by the use of the integrated momentum equations. The velocity distribution

is approximated by a sextic polynomial. Numerical results are given for several different values of $\mu_2\rho_2/\mu_1\rho_1$ for $U_2/U_1=10$. Here μ is the viscosity, ρ is the density, U is the free stream velocity and subscripts 2 and 1 refer to the lower and upper stream respectively. *R. C. Di Prima.*

Rubesin, Morris W.; and Inouye, Mamoru. A theoretical study of the effect of upstream transpiration cooling on the heat-transfer and skin-friction characteristics of a compressible, laminar boundary layer. NACA Tech. Note no. **3969** (1957), 41 pp.

Consider a laminar, compressible flow over a semi-infinite flat plate. The front portion of the plate is porous and is transpiration cooled. The effects of this transpiration cooling on the rest of the boundary layer are investigated using the momentum integral technique with a velocity profile approximated by a seventh-degree polynomial. The inhomogeneous energy equation is solved by expressing the total energy in terms of the velocity (Prandtl number=1). To this solution are added solutions of the homogeneous energy equation to satisfy the boundary conditions on the temperature distribution. The case in which the heat transfer at the wall is prescribed is also considered.

One of the conclusions that is reached is that when the total coolant rate is kept constant and the porous length is varied, the average skin friction on a partially porous plate is lower than on a fully porous plate. The local heat transfer behaves in a manner similar to that of the local skin friction. *R. C. Di Prima* (Troy, N.Y.).

Chandrasekhar, S.; and Donnelly, R. J. The hydrodynamic stability of helium II between rotating cylinders. I. Proc. Roy. Soc. London. Ser. A. **241** (1957), 9–28.

The writers investigate the hydrodynamic instability of the two-fluid model of liquid helium II below the λ-point between rotating cylinders. Some conditions are assumed regarding the mutual friction force between the normal and the superfluid components of the liquid. This force is proportional to the constant vorticity and to the difference in the velocities between both fluids. The problem of hydrodynamic stability is solved when the two cylinders rotate in the same direction. The flow becomes unstable along two branches. One is the well-known Rayleigh instability of the inviscid superfluid; the second is the well-known Taylor instability of the viscous liquid. There exists a coupling between the two fluids (viscous and inviscid) and the calculated value of the coupling constant agrees well with the value of this constant deduced from the experiment of H. Hall and W. F. Vinen [same Proc. **238** (1956), 204–214, 215–234].

M. Z. v. Krzywoblocki (Urbana, Ill.).

Chandrasekhar, S. The hydrodynamic stability of helium II between rotating cylinders. II. Proc. Roy. Soc. London. Ser. A. **241** (1957), 29–36.

The problem of the hydrodynamic instability of the two-fluid model of helium II leads to some characteristic value problems solved in the present paper. Moreover, the nature of the two kinds of instabilities, discussed above, is further discussed here. *M. Z. v. Krzywoblocki.*

Favre, A. J.; Gaviglio, J. J.; and Dumas, R. Space-time double correlations and spectra in a turbulent boundary layer. J. Fluid Mech. **2** (1957), 313–341.

This paper describes detailed measurements of velocity fluctuations in a turbulent boundary layer on a flat plate. The measurements, which concern only the velocity component parallel to the mean stream, include the first easily accessible study of the space-time double-velocity correlation, i.e. the correlation coefficient between the velocity fluctuation at one point and that at another point subject to a fixed time delay. From the numerous and extremely concentrated measurements, a few results may be selected for notice: (a) Outside the conventional "edge" of the layer, the spectrum of the velocity fluctuations shows a marked maximum, believed to be characteristic of the irrotational motion induced by movements of the boundary to the turbulent fluid. (b) For a fixed separation perpendicular to the surface, the space-time correlation has a maximum at a non-zero delay time which increases with separation. (c) A similar optimum delay time exists for separations with components both perpendicular to the surface and parallel to the free stream, and this delay is compounded of the delay time appropriate to the normal separation and the time of flight over the longitudinal separation. (d) Lines drawn through points of maximum space-time correlation for a fixed longitudinal separation run outwards, across the mean streamlines, and along them high correlation is preserved for considerable separations. *A. A. Townsend* (Cambridge, England).

Hakkinen, Raimo J.; and Richardson, A. S., Jr. Theoretical and experimental investigation of random gust loads. I. Aerodynamic transfer function of a simple wing configuration in incompressible flow. NACA Tech. Note no. **3878** (1957), 64 pp.

This is a report on experiments and calculations relating to the lift on a wing in turbulent air at relatively low speeds. The experiments were carried out in a wind tunnel with a constant-chord wing spanning it, the input gusts being provided by (a) another airfoil, upstream, carrying out sinusoidal oscillations, or (b) a coarse square-mesh grid. In case (a), according to oscillating-airfoil theory, the downwash velocities in the wake of the upstream airfoil are sinusoidal [H. Ashley, Thesis, Mass. Inst. Tech., 1951], and the lift produced on the stationary, rigid, downstream airfoil is also sinusoidal and given by the twodimensional sinusoidal-gust function [W. R. Sears, J. Aero. Sci. **8** (1941), 104–108; MR 3, 74; also Thesis, Calif. Inst. Tech., 1938]. There were experimental difficulties, and it was not possible to measure the phase relationship of lift and gusts; nevertheless, the agreement between measured and theoretical lift amplitude over a modest range of reduced frequencies (0.08 to 0.24) seems satisfactory.

An expression for the lift developed on the wing in a random gust field is worked out, following essentially the procedure of H. W. Liepmann [J. Aero. Sci. **22** (1955), 197–200]. This leads to an expression for the power spectrum of lift in terms of the power spectrum of the turbulence, involving a two-wave-number transfer function instead of the two-dimensional gust function discussed above. For lack of more theoretical information, it is assumed that "strip theory" applies and that the lift of each strip follows the two-dimensional function. This theory indicates how the mean-square lift is affected by changes of span. It also permits a further experimental determination of the function after measurement of the two power spectra. Again, the agreement between experiment and theory seems encouraging, especially at reduced frequencies greater than 1. *W. R. Sears.*

★**Kampé de Fériet, J. Problèmes mathématiques posés par la mécanique statistique de la turbulence.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. **III**, pp. 237–242. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Soit R un anneau de fonctions à valeurs réelles $f(x)$ sur un ensemble X, contenant les constantes αC_X, où C_X est la fonction caractéristique de X. En vue d'obtenir une définition consistante de la notion de moyenne d'une fonction, l'auteur introduit des applications linéaires Tf de R sur lui même, qu'il appelle transformations de Reynolds, et qui ont les propriétés suivantes: i) $T(fTg)=TfTg$; ii) Si $V(f)$ est un voisinage de V suivant une topologie donnée, $g\in V(f)$ entraîne $Tg\in V(Tf)$; iii) $TC_X=C_X$. Se donnant ensuite une σ algèbre $\mathfrak{F}$ de parties de X, il étudie les transformations de Reynolds dans l'anneau R des fonctions mesurables par rapport à F. Il passe en revue les propriétés déjà connues de ces transformations et démontre un théorème nouveau permettant d'en construire effectivement. *J. Bass* (Paris).

Ludford, G. S. S.; and Schot, S. H. On sonic limit lines in the hodograph method. Math. Z. **67** (1957), 229–237.

The authors show that solutions of the equations of steady two-dimensional, irrotational, isentropic flow of a perfect gas, with constant adiabatic index γ, obtained by means of the hodograph transformation, can possess, in a much wider range of cases than had previously been suspected, a "sonic limit line". This means that, in the physical plane, streamlines approaching the sonic line from the supersonic side turn back from it "onto another Riemann sheet", and exhibit cusps on the sonic line. This was known previously only for the "source flow" $\psi=k\Theta$ (where ψ=stream function, Θ=direction of flow), but is now shown to be true also for any linear combination of solutions $\psi=F_n(\tau)\cos n\theta$ or $F_n(\tau)\sin n\theta$, where $F_n(\tau)$ is a solution of a familiar hypergeometric equation, selected so that $F_n'(\tau)=0$ for the sonic value $(\gamma-1)/(\gamma+1)$ of τ. The characteristics are tangential to such a sonic line (as in another rather special problem, that of linearized conical-field solutions) instead of having cusps on it as in more general cases.

It is necessary to emphasize, however, that although a fairly wide class of solutions obtainable by the hodograph method has thus been shown to be of no value, there remains an even wider class of solutions which do retain some physical significance. *M. J. Lighthill.*

Mazelsky, Bernard. Theoretical aerodynamic properties of vanishing aspect ratio harmonically oscillating rigid airfoils in a compressible medium. J. Aero. Sci. **23** (1956), 639–652.

Assuming that the velocity potential satisfies the two-dimensional wave equation in planes transverse to the flight axis (high frequency, slender wing approximation), the author introduces elliptic coordinates and obtains solutions in Mathieu functions for the rigid body motions of rectangular and triangular wings. He cites the parallel work of Merbt and Landahl [Roy. Inst. Tech., Div. Aero., Stockholm, KTH Aero. TN **30** (1953)] but not their later, more extensive results [ibid. **31** (1954)], nor those of Milne [Coll. Aero. Cranfield. Rep. no. **94** (1955); MR **17**, 914]; nor those of the reviewer [Miles, Aero. Quart. **4** (1953), 231–244; MR **15**, 367; Winograd and Miles; ibid. **7** (1956), 106–108]. He also appears unaware of the analogous work on the diffraction problem, initiated by B. Sieger [Ann. Physik (4) **27** (1908), 626–664] and continued by many others. His numerical results are, however, more complete than any available heretofore. *J. W. Miles.*

Kondo, Kazuo. On the potential-theoretical fundamentals of the aerodynamics of screw propellers at high speed. J. Fac. Engrg. Univ. Tokyo **25** (1957), 1–39.

In der vorliegenden Arbeit, die bereits vor mehr als 10 Jahren abgeschlossen wurde, werden die grundlegenden Beziehungen für die Berechnung schwach belasteter Propeller bei reibungsloser, kompressibler Strömung — unter Beschränkung auf den Unterschallbereich — hergeleitet. Hierbei wird nicht, wie allgemein üblich, der Propeller durch ein System tragender Wirbel ersetzt, sondern das von L. Prandtl [Z. Angew. Math. Mech. **16** (1936), 360–361] vorgeschlagene Beschleunigungspotential verwendet. Bei der Propelleruntersuchung werden dann die tragenden Flächen durch eine Verteilung von Dipolen und Quellen ersetzt. Mit den sich ergebenden grundlegenden Beziehungen wird abschliessend gezeigt, dass die von A. Betz [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1919**, 193–217] für inkompressible Propellerströmung angegebenen Theoreme auch bei kompressibler Strömung unverändert gültig bleiben. *L. Speidel.*

Morawetz, Cathleen S. On the non-existence of continuous transonic flows past profiles. II. Comm. Pure Appl. Math. **10** (1957), 107–131.

Cet article apporte un important complément à l'étude présentée dans la partie I [mêmes Comm. **9** (1956), 45–68; MR **17**, 1149]. On considère un écoulement continu irrotationnel transsonique symmétrique autour d'un profil symmétrique P_0. Le nombre de Mach à l'infini est inférieur à 1, il existe une région finie où l'écoulement est supersonique. On considère un profil P qui ne diffère de P_0 que le long d'un arc de P_0 découpé par deux caractéristiques concourrantes. Existe-t-il un écoulement continu autour de P pour le même nombre de Mach à l'infini? Dans la partie I, l'auteur avait prouvé que, dans le cadre de la théorie des perturbations où tout terme quadratique par rapport aux différences des vitesses et des accélérations est systématiquement négligé, il existe au plus une famille dépendant d'un paramètre de tels profils P. Dans le présent article les équations exactes sont utilisées et il est prouvé qu'il n'existe aucun profil P donnant lieu à un écoulement continu. Le résultat est atteint sous des hypothèses assez larges (différences des vitesses et des accélérations bornées). La méthode consiste à formuler dans le plan de l'hodographe l'équation et les conditions aux limites vérifiées par la différence des potentiels. Essentiellement, et c'est là le point difficile et délicat du travail, il s'agit de prouver un théorème d'unicité montrant que cette différence est nécessairement constante.
P. Germain (Paris).

Coupry, Gabriel. Flottement de gouverne à un degré de liberté en écoulement transsonique. C. R. Acad. Sci. Paris **244** (1957), 2350–2352.

This is a study of control surface oscillations under transonic conditions. Beginning with the equations of unsteady transonic flow, with the customary approximations, the author next writes down the linearised equation for an unsteady perturbation potential which is superimposed on a steady field. Introducing further simplifications and special assumptions, he distinguishes three cases according as the shock wave is situated up-

stream of, on, or at the trailing edge of, the control surface. The author concludes that there is a tendency towards instability in the second, but not in the third case, while the first case is intractable, except for an observation of a very general character. {This paper is of a predominantly non-mathematical character. It is frankly speculative but may be of interest when taken in conjunction with empirical evidence.} *A. Robinson.*

Hunter-Tod, J. H. The aerodynamic derivatives with respect to a rate of yaw for a delta wing with dihedral and at incidence at supersonic speeds. Aero. Res. Council, Rep. and Memo. no. **2887** (1951), 23 pp. (1957).

The author uses linearised aerodynamic theory to obtain the forces acting on a delta wing which is travelling at supersonic speeds and undergoing pure yawing motion. The use of linearised theory is only valid when both the incidence and dihedral are small. The author also assumes that both leading edges are either subsonic or supersonic. The results for these two cases are given analytically and graphically. The derivatives, for the case of subsonic leading edges, are compared with similar results obtained using strip-theory. The comparison shows that the values of the derivatives obtained in this paper are less than the strip theory values. (The former are certainly more accurate than the latter.) The percentage error is a function of aspect ratio and the agreement is good for small aspect ratios but poor otherwise. *G. N. Lance* (Southampton).

v. Krzywoblocki, M. Z. On axially symmetric flow through annular bodies in compressible flow. Bull. Tech. Univ. Istanbul **10** (1957), 60–83. (Turkish summary)

In the design of the present day aircraft the aerodynamics of open-nosed slender body is often of considerable importance. Weinstein, Sadowsky, Sternberg, Streeter and others treated various cases of a flow of such a nature referring to an incompressible fluid. The present paper presents a linearized method of solving the analogous problem in the domain of a compressible fluid flow. The method is illustrated by means of an example.

Author's summary.

Meyer, R. E. On the measurement of supersonic aerofoil drag by pressure traverse. Aero. Quart. **8** (1957), 123–144.

Two-dimensional flow about an airfoil spanning a supersonic wind-tunnel stream is considered. A formula for drag is written down from momentum principles, and it is proposed to evaluate this experimentally by measurement of static and stagnation pressures, as is often done in low-speed flow. Approximate expressions are worked out for the integrand of the drag-coefficient integral for various portions of the control contour, and the errors involved are discussed in detail. Several difficulties are pointed out, including the problem of ascertaining shock-wave positions to the required accuracy, and the serious effect of "displacement effect" of pitot-tube readings in the viscous wake. The catalog of possible sources of error is so extensive that it is somewhat surprising that the author concludes that the pessimism of Holder, MacPhail and Thompson [Modern developments in fluid dynamics ···, v. 2, L. Howarth, ed., Oxford, 1953, pp. 567–568; MR **15**, 574] is not confirmed. *W. R. Sears.*

Carter, W. J. Optimum nose shapes for missiles in the superaerodynamic region. J. Aero. Sci. **24** (1957), 527–532.

The meridian profile for the body of revolution having minimum drag (for given fineness ratio) in free molecular flow is established through the calculus of variations assuming either specular (also called "spectral" by the author) reflection, with and without roughness, or random reflection. The reviewer questions whether the author's basic assumptions are sufficiently realistic to lead to physically significant results. *J. W. Miles.*

Stetter, H. J. Eine Verallgemeinerung des Kármán-Moore-Verfahrens zur Berechnung der linearisierten Überschallströmung um schlanke Drehkörper. Z. Angew. Math. Mech. **37** (1957), 145–146.

This note proposes an improved version of the numerical procedure of von Kármán and Moore [Trans. A.S.M.E. **54** (1932), APM, 303–310] and Tsien [J. Aero. Sci. **5** (1938), 480–483] for calculating linearized supersonic flows past bodies of revolution. *Y. H. Kuo* (Peking).

Freeman, N. C. On the stability of plane shock waves. J. Fluid Mech. **2** (1957), 397–411.

In a previous paper [Proc. Roy. Soc. London. Ser. A. **228** (1955), 341–362; MR **16**, 878] the author investigated the "stability" of shape of plane shock waves by a theoretical treatment of how the shock wave, produced by a not perfectly plane piston moved impulsively into fluid at rest, would tend to a plane shape. In the present paper the stability is studied by a theory of a more physically realizable flow; once more, the squares of disturbances to a uniform flow behind a plane shock wave are neglected. Here, a plane shock wave passes along a channel whose walls are parallel except in a certain limited stretch where they depart slightly from the plane parallel shape. The shock wave is perturbed and the return to the plane shape is again studied. Once more, the greatest stability is indicated by the theory to occur for a shock wave Mach number of 1.15, and strong shocks appear to be considerably less stable. For shocks that are not strong, perturbations are predicted to decay as the $(-\frac{1}{2})$th power of the distance travelled.

For a roof-top-shaped disturbance to the wall of the tube, the theoretical predictions are displayed in detail and compared with experiments by Lapworth (to be published); three is agreement on some points; disagreement on others is ascribed principally to non-linear effects of the kind discussed in the recent paper of Whitham [J. Fluid Mech. **2** (1957), 145–171; MR **19**, 206].

M. J. Lighthill (Manchester).

Alpher, Ralph A. The Saha equation and the adiabatic exponent in shock wave calculations. J. Fluid Mech. **2** (1957), 123–126.

A short review is given of methods for calculating shock transitions where ionization phenomena occur. Possible errors are pointed out and references are given which should be helpful in performing such calculations.

C. S. Morawetz (New York, N.Y.).

Smith, P. W., Jr. Sound transmission through thin cylindrical shells. J. Acoust. Soc. Amer. **29** (1957), 721–729.

The effect of a thin cylindrical shell on a sound wave is solved approximately. The author evaluates the impedance at the surface of the cylinder whose form he approximates by considering the radial component only. Examples are evaluated and discussed. *H. Feshbach.*

Heaps, H. S. Reflection of plane waves of sound from a sinusoidal surface. J. Appl. Phys. **28** (1957), 815–818.

The author calculates an approximation to the reflection of a plane wave from the pressure-release surface $z=c\cos px$ by including only undamped, outgoing plane waves but without restricting c or p. He concludes that further improvement must be based on the inclusion of other forms of radiation than undamped plane waves in $z<c\cos px$; it is not clear to the reviewer whether and/or why he thereby excludes the possibility of undamped, incoming plane waves in $z<c\cos px$. *J. W. Miles.*

Brillouin, J. Reflection and refraction of acoustic waves by a shock wave. NACA Tech. Memo. no. 1409 (1957), 42 pp.

A translation from the French of Acustica **5** (1955), 149–163; MR **17**, 208.

Hochstadt, Harry. Asymptotic formulas for diffraction by parabolic surfaces. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. EM–89 (1956), i+31 pp.

In this report the author has treated the problem of reflection of plane waves (acoustic) falling on the concave side of a paraboloid of revolution (p.r) and also on a parabolic cylinder (p.c.). The following cases have been treated: (a) reflection of a spherical wave from a point source located at the focus and on the axis of the p.r., and (b) a line source along the focal line or located at some distance and parallel to the focal line for the p.c., when the wave number $k=2\pi/\lambda$ is large. After deriving the caustics, the author has investigated the possible stationary points of the distance from the source to the observation point located away on the far side of the focal surface (caustic), on the caustic and in the region between the caustic and the reflecting surface. The knowledge of the stationary points is necessary for the evaluation of the field at the observation point (o.p.) when k is large. After these prelimenary results the author has obtained expressions for the Green's functions of the p.r. and p.c. for both the Dirichlet and Neumann problems, with the aid of which the reflecting fields are expressed in the usual way. To obtain explicit expressions for the field at the o.p. for the various cases mentioned above the method of stationary phase is employed. For the case of a plane wave falling on the p.r., the leading term is found to be independent of k. It contains two terms arising from two stationary points located symmetrically with respect to the reflecting surface. If the source is located on the axis, then only one term enters into the leading term of the field expression. Analogous expressions are found for the case of reflection from the p.c., with no dependence on k. If the source is located at the focus, the leading term is found to be independent of k but for the p.c. the leading term contains a factor of $k^{-\frac{1}{2}}$. For a point source located on the axis but not at the focus the leading term varies as $k^{\frac{1}{2}}$ in the p.r. case and for a line source of the focal line when the parabolic coordinate $\eta=0$, i.e. when the o.p. is on the axis. For an observation point within the caustic and the reflecting surface, no stationary point exists and the value of the field is found to be independent of k (leading term). The behavior at the caustic and at the cusp is found to vary as $k^{1/6}$ with different coefficients. The behavior of the field at the cusp depends on the way it is approached from outside or from within the caustic and the reflecting surface. For a line source located on the axis of the p.c. the leading term of the field varies as $k^{-\frac{1}{2}}$, whereas at the cusp when approached from within, the expression for the field has the same k dependence but a different coefficient. For points on the axis and in the vicinity of the cusp, the field behaves as $k^{-1/4}$ and remains finite. If the cusp is approached along the caustic, one finds a $k^{-1/3}$ dependence. Finally, in the vicinity of the cusp, a similar dependence of the field is obtained, and the dependence of the field on the coordinate ξ is like $\xi^{-1/6}$ on the caustic and like $\xi^{-\frac{1}{2}}$ on the axis of the cylinder.

N. Chako (Flushing, N.Y.).

Maidanik, G. Acoustical radiation pressure due to incident plane progressive waves on spherical objects. J. Acoust. Soc. Amer. **29** (1957), 738–742.

The force acting on a sphere with complex index of refraction struck by a plane sound wave is obtained in terms of the constants required to evaluate the scattering of the sound wave. The result is evaluated for hard and soft spheres. *H. Feshbach* (Cambridge, Mass.).

Stix, Thomas H. Oscillations of a cylindrical plasma. Phys. Rev. (2) **106** (1957), 1146–1150.

The equilibrium state of an infinitely long cylinder of plasma of finite density and zero pressure, immersed in a vacuum and subject to a uniform axial magnetic field, is perturbed on the assumption of infinite conductivity and negligible gravity. Oscillations at frequencies small compared to the electron plasma and electron cyclotron frequencies are considered. In an analysis deriving from Kruskal and Schwarzschild [Proc. Roy. Soc. London. Ser. A. **223** (1954), 348–360; MR **15**, 914] the Maxwell equations are supplemented by the magnetohydrodynamic equations

$$\rho\frac{\partial\vec{v}}{\partial t}=\vec{j}\times\vec{B},\qquad \vec{E}+\frac{\vec{v}}{c}\times\vec{B}-\frac{1}{en_e}\vec{j}\times\vec{B}=0.$$

and boundary conditions follow by integrating across a thin layer of plasma. Cylindrically symmetric perturbations yield a dispersion relation which indicates a natural frequency of oscillation that approaches the ion cyclotron frequency as the wave-length in the z-direction becomes shorter. Near the resonant frequency the radial and the azimuthal electric field are of the same order of magnitude, the plasma ions rotate around the magnetic field, and the space charge due to the ionic motion is neutralized by electrons flowing along the magnetic field.

C. D. Calsoyas (Livermore, Calif.).

Stewartson, K. Magneto-hydrodynamics of a finite rotating disk. Quart. J. Mech. Appl. Math. **10** (1957), 137–147.

An attempt is made to account for certain observations of Lehnert by considering the flow within an infinite shallow horizontal layer of mercury in a vertical magnetic field when part of the base is set in rotation.

For a large magnetic field a steady solution is found wherein the mercury above the disk rotates bodily up to the edge of the disk while it is at rest in the region beyond. Above the edge there is a thin friction layer separating the two regions, above which the free surface is parabolic and plane.

Although such a solution is in agreement with Lehnert's observations, it is found that for the theory to be applicable the magnetic field should be considerably greater than the highest value he used. Accordingly, the problem is considered afresh with the magnetic field absent. A

qualitively similar steady solution is derived numerically. Since it is known on experimental grounds that the flow in the friction layer is likely to be unstable, and on theoretical grounds that magnetic fields tend to stabilize such motions, it is inferred that the motions observed were in accord with the field-free theory but that the magnetic fields were necessary to ensure stability.

K. C. Westfold (Pasadena, Calif.).

Abbi, S. S.; and Chandra, R. On the equilibrium of a small conducting liquid drop in a uniform external electric field. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 363–368.

Conditions for equilibrium are sought when electrostatic attraction and surface tension forces are considered. A two-parameter family of surfaces representing possible deformations from the spherical shape is assumed. It is shown to be possible to find values for the parameters corresponding to equilibrium when the drop is sufficiently small. The resulting figure is a prolate spheroid with long axis in the field direction. *E. Pinney.*

Kidder, Ray E. Motion of the interface between two immiscible liquids of unequal density in a porous solid. J. Appl. Phys. **27** (1956), 1546–1548.

From the author's summary: A general solution is obtained to a two-dimensional problem of the movement of the interface between two immiscible liquids of unequal density in a porous solid, the interface motion being the result of the force of gravity acting on the two liquids.

J. J. Stoker (White Plains, N.Y.).

See also: Computing Machines: Kovásznay. **Elasticity, Plasticity:** Ziegler. **Structure of Matter:** Mattis. **Optics, Electromagnetic Theory, Circuits:** Boillet. **Classical Thermodynamics, Heat Transfer:** Markstein. **Astronomy:** Zeuli. **Geophysics:** Monin; Krastanov.

Optics, Electromagnetic Theory, Circuits

Köhler, H. Bemerkungen über die Korrektion der Verzeichnung bei Fernrohren. Optik **14** (1957), 241–255.

An old problem in the theory of telescopic optical systems is whether the distortion should be corrected so that $\tan w'/\tan w = \text{const}$, or $w'/w = \text{const}$, or according to some other condition, for best visual use. This question was reexamined both experimentally (with commercial binoculars) and theoretically. In the case of a fixed telescope, the tangent condition was found to be necessary to preserve straight lines. For a moving telescope a "buckling" effect in the image was observed, and the angle correction was found to correct this effect. However, the tangent-type and angle-type corrections cannot both be applied in the same instrument. *E. W. Marchand.*

Schaefer, W. Zur Korrektur von Ablenkfehlern bei einer Prismenanordnung für einen Heliostaten. Optik **14** (1957), 221–229.

By optical considerations it is found possible to simplify the construction of a certain type of photometer used in measuring the glide angle of glider planes. A double-prism arrangement may be used in place of a variable mirror device. *E. W. Marchand.*

Köhler, H. Zusammenhänge zwischen der ausseraxialen Zerstreuungsfigur und dem axialen Korrektionszustand optischer Instrumente. Optik **14** (1957), 202–220.

Methods are described for improving the off-axis correction of telescopes at the expense of the axial correction so as to obtain a better over-all result. The Seidel (third-order) theory is used. *E. W. Marchand.*

Flügge, J. Zur Theorie der astigmatischen Zweistärkengläser. Optik **14** (1957), 193–201.

Rules are derived for locating the optical center of the lower part of astigmatic bifocal glasses. Also the values of the prismatic deviation are studied for various regions of the reading segment. *E. W. Marchand.*

Linfoot, E. H. Image quality and optical resolution. Opt. Acta **4** (1957), 12–16.

The author discusses figures of merit for an optical system. He enumerates the objections to the use of resolving power, the chief one being that it is based on knowledge of the object (two separated bright stars). He discusses definition, based on statistical concepts, and suggests the use of a quantity called by him, "structural resolution", which he describes as the ability to discern low contrast objects through the photon noise. The quantity for which a statistical mathematical formulation is given, is formally identical with proposals by Fellgett [J. Opt. Soc. Amer. **43** (1953), 271–282] and Schade [J. Soc. Motion Picture Television Engrs. **58** (1952), 181–222]. The author deduces from this theory that the image of a low contrast object deteriorates significantly if the structural resolution as thus defined then deteriorates by 20%. This quantity could be used as giving a tolerance for the image quality of an optical system, either due to image errors or due to diffraction errors occurring in stopping down the system. *M. Herzberger* (Rochester, N.Y.).

Le Corre, Yves. Théorie tensorielle du pouvoir rotatoire magnétique. J. Phys. Radium (8) **18** (1957), 312–317.

Die elektrische, bzw. magnetische Feldstärke wird zunächst durch allgemeine lineare Gleichungen mit der elektrischen und magnetischen Induktion verbunden. Hierdurch werden Hermitesche Matrizen definiert. Je nach der Symmetrieklasse des Mediums werden diese spezialisiert, was in Tabellenform angegeben ist. *J. J. Burckhardt.*

Silberg, Paul A. Infrared absorption of three-layer films. J. Opt. Soc. Amer. **47** (1957), 575–578.

The absorption of a film consisting of a dielectric layer sandwiched between two metals is calculated. Absorption exceeding 95% is obtainable. This result is a generalization of a result of C. Hilsum [same J. **44** (1954), 188–191], who assumes identical parameters for the entrance and exit film. *A. A. Blank* (New York, N.Y.).

Hopkins, H. H. The numerical evaluation of the frequency response of optical systems. Proc. Phys. Soc. Sect. B. **70** (1957), 1002–1005.

Greiner, H.; und Schäffer, E. Zur Seyaschen Theorie des Konkavgitter-Spektrometers. Optik **14** (1957), 263–276.

After a brief discussion of various types of single reflecting grating spectrometers (monochromators) the author has analyzed the essential features of the so-called Seya's theory of the concave grating spectrometer [Science of Light **2** (1952), 8–17]. The theory is based, on the

one hand, on the interference relation between the wavelength λ, the angles of incedence and reflection α, β of the light beam and the grating distance d and, on the other hand, on the focusing condition relating the distances r and r' of the entrance and exit slits, measured from the vertex point S of the grating, the radius of the Rowland circle R, the lateral distance (displacement) w measured from S and α and β. In this theory the second and higher order terms in the ratio w/R are neglected, thus reducing the focusing condition to a simple equation in α, β, r, r' and R. Approximate solutions of this equation are sought, for fixed r and $C=\alpha-\beta$, when the grating is rotated about the normal to the grating passing through S so that one gets a minimum displacement $\Delta r'$. By expanding the quantities in the first order equation (Taylor series about α_0 (fixed incident angle)) and retaining terms up to $(\Delta\alpha_0)^2$, a linear homogeneous system in $1/r$, $1/r'$, and $1/R$ is obtained, which leads to a third degree equation in C as function of α_0. From the condition $dC/d\alpha=0$ and the auxiliary conditions $dr/d\alpha_0=dr'/d\alpha_0=0$ for $C=$const. and for $dr'/d\alpha_0=0$ for r and $C=$constants, one gets an approximate value of $C=70°15'$ with $r/R=1.22247$. In section three the author discusses Seya's solution for C which is derived in the previous section. By taking account of the first two auxiliary conditions the author has obtained relations between the ratios R/r and R/r' as functions of α and C. Several curves are plotted for these ratios against α $(0<\alpha<90°)$ for different values of C ranging from 69° to 70°5′ for positive and negative order. It is found that for $\alpha=\pm 9°$, i.e. for a wave length range of 0—4300 Å, the value of C satisfying these relations is 70°15′. By taking into account the third auxiliary condition the author has calculated favorable values of C for various spectral ranges, different values of α and of the ratios R/r, R/r' for fixed values of $R=1000$ mm and the grating distance $d=1/600$ mm. Furthermore if the slits do not lie on the Rowland circle, the second and higher order terms in R/r' for fixed α, β, R/r and w contribute to the displacement of r'. As a result, an inequality between these quantities and the wavelength of light based on Rayleigh's criterion for best resolution is derived. *N. Chako.*

Giovanelli, R. G. Diffused radiation in semi-infinite and thick plane parallel diffusers with a line source parallel to the surface. Opt. Acta 3 (1956), 49–55.

In this paper the author has considered the problem of finding the total intensity of diffused radiation inside or on the surface of a semi-infinite extension of a thick slab of parallel layers due to a line source parallel to the slab surface and located in or on the surface of the diffusing medium. By separating the total diffused intensity into several parts, namely, (a) that due to diffused radiation from the semi-infinite medium A; (b) the intensity of diffused radiation due to radiation which would escape A and would enter a second medium B, adjoining A, this being equivalent to a source distribution separating A and B, and (c) the direct diffused radiation from the source (unscattered and/or unabsorbed). The author has derived expressions for each part contributing to the total intensity of diffused radiation and hence has found an expression for part (a) in terms of the others. The function representing (a) has been computed for two cases where the ratio of the isotropic scattering coefficient to the extinction coefficient is very close to unity. For a very thick slab bounded with parallel planes (infinite extent) containing a line source parallel to the surface, the author has derived an expression for part (a) of the diffused intensity radiation in terms of the total intensity of diffused radiation and in terms of other contributions analogous to part (b) and (c) mentioned above.

The author points out that for a more accurate estimation of (a) one has to solve an integral equation for the function representing the intensity of diffused radiation in part (a). It is assumed throughout the paper that the diffused radiation is isotropic. *N. Chako.*

Ingarden, Roman Stanisław. On the geometrically absolute optical representation in the electron microscope. Trav. Soc. Sci. Lett. Wrocław. Ser. B. no. **45** (1957), 60 pp.

This paper is concerned with the problem of absolute image formation by an electron optical system. The question which the author has raised is whether there exists an absolute image representation of a three dimensional region in electron optics. To answer this question the author starts his analysis by giving precise definitions of absolute, perfect or ideal, and stigmatic image representations of a set of points in the object space, and by defining first caustical points as those points P in the field, which are reached by more than one of the electrons issuing from a point P_0 at the same instant t_0. A point P is a strong caustical point or caustic if a continuum of trajectories pass through it, a weak one if only a finite or denumerable number pass through it, and a quasi-one if it belongs to the set of limit points of caustical points not included in the set. If a point P_1 is a strong caustic point of some point P_0 and time t_0, then P_1 is an absolute optical image of P_0, and if the solid angle at P_1 is 4π, it is called a complete absolute image of P_0. If the particles issuing from P_0 at t_0 reach P_1 at the same instant, P_1 is called a maximal absolute image. Ideal or perfect images are then defined as maximal absolute. By a stigmatic point it is meant that P_1 is a strong caustical point of a caustic of P_0 and isolated from other caustical points of P_0 at t_0. Therefore, absolute image points are stigmatic but not vice-versa. These are illustrated by two examples where the refractive index varies inversely as the distance and a generalized form of the Maxwell's fish eye.

In order to answer the question raised in this paper, the author has first investigated the structure of Finsler space, since the Lagrangian in electron optics depends both on the coordinates and directions (the space is anisotropic). In the second chapter one finds a detailed analysis of Finsler spaces, including expressions for the point mixed and angular curvature tensors and the torsion tensor. By transforming the Lagrangian (relativistic) the author obtains an expression for the distance element in the form (i) $ds=\varphi_i(x^i)(\delta_{ij}dx^i dx^j)^{\frac{1}{2}}+a_i(x^i)dx^i$ [$\varphi_i(x^i)$, $a_i(x^i)$ are dependent of the fields] for a Lorentz system, from which the direction and torsion vectors and torsion tensor are calculated. In the third chapter the author begins with a detailed study of Maxwell's fish eye following Caratheodory's analysis. This leads to the imbedding of the three-dimensional Riemann space into a four-dimensional spherical surface. This medium gives an absolute representation, not only of points but of the whole three-dimensional space. It is followed by an analysis of constructing a medium (for electron optical systems) possessing the properties of such a surface as in Maxwell's fish eye, in which case one could find a perfect electron optical system for the whole ordinary space. Since the ordinary optical medium of the fish eye could be considered as a Riemann space with constant curvature, the author has shown that an electron optical medium violates this

property. This does not, however prove the impossibility of perfect representation of ordinary space by electron optical instruments. The author believes that such a representation is possible. To prove this he assumes the validity of a theorem of Blaschke [Vorlesungen über Differentialgeometrie, vol. I, 3rd ed., Springer, Berlin, 1930, p. 228; Abh. Math. Sem. Hamburg. Univ. **11** (1936), 409–412] in somewhat modified form. This leads to the problem of imbedding a Finsler space into a Minkowskian space of twice its dimensions. A special and explicit solution of this problem has been given by the author [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 305–308; MR **16**, 173]. The next two sections deal with the analysis of the electrostatic and electromagnetic microscope where it is shown that the assumption of a constant curvature of the respective Riemann spaces is in contradiction to Maxwell's equations.

The last chapter is concerned with the absolute representation of a single point for velocity modulated and other type of electron optical systems. Here, as well as in previous chapters, diffraction and aberration effects are neglected. The problem is to find a solution of the wave equation inside the cylinder $r^2=x^2+y^2+z^2\leq 1$ $(0\leq t\leq\infty)$ such that $u|_{t=0}=\partial u/\partial t|_{t=0}=0$ and $u|_{r=1}=G(t,\vartheta,\phi)$. The problem is solved for the case where G is independent of ϕ and where the Lorentz term arising from the magnetic field is small in comparison to the electric force, or interpreted in another way, where relativistic effects are neglected, since the forces due to the magnetic field are of the order v^2/c^2. In such cases an absolute imaging of a single point is possible for non-static fields.

Finally, the main questions of whether an absolute or stigmatic electron optical instrument imaging all space exists is left unanswered. *N. Chako.*

Paschke, Fritz. Die Wechselseitigkeit der Kopplung in Wanderfeldröhren. Arch. Elek. Übertr. **11** (1957), 137–145.

The relations giving the mutuality of coupling between two lines [S. E. Miller, Bell System Tech. J. **33** (1954), 661–719] are applied to the traveling wave tube. In the traveling wave tube case, the law states that both a field of the undisturbed circuit wave on the beam and a field of the undisturbed space charge wave on the delay line have to be present in order to obtain a coupled wave and amplification. The coupling is that from the circuit to the electron beam given by the coupling parameter of the delay line $K=|\bar{E}^*|\cdot|E|/2P\cdot\beta^2$, and from the beam to the circuit given by the expression $(1-q^2)$, where $q=\omega_p'/\omega_p$ is the reduction of the plasma-frequency by the delay line [G. M. Branch and T. G. Mihran, I.R.E. Trans. **ED-2** (1955), no. 2, 3–11].

Under the following conditions: 1) one dimensional ion-neutralized electron stream with local constant d.c.-density and uniform velocity; 2) small signals; 3) phase velocities small compared to the speed of light; 4) axial-electrical a.c.-field in the beam cross-section approximately constant; the gain-parameter C, the space charge parameter Q, and the phase velocity parameters were determined by means of the Pierce theory in an earlier paper by the author [Frequenz **9** (1955), 273–279].

In the present paper, the convection current and the displacement current of the electron beam are considered. In order to employ the curves and tables of Pierce [Traveling-wave tubes, Van Nostrand, New York 1950], the author gives the necessary transformation equations. It is found that the field distribution in the electron beam $E/(\omega t)$ of the T.W.T. shows better approximation to the field distribution of the free space charge wave than to the field distribution of the delay line. The equations derived in the present paper give rules of design for low-noise T.W.T. amplifiers. The analysis shows that 1) the beam diameter has to be large, 2) the beam velocity has to be low, and 3) the coupling parameter has to be large.

This means physically that for constant amplifying properties it is required that C, $C\cdot Q$, and thus $Q\cdot C^3$ must be constant. However, the reduction of $(1-q^2)$ in a sense of lowering the noise coupling from the beam to the circuit requires an increase of $Q\cdot C^3$. In consequence, it follows that a larger part of the noise energy is coupled into the non-amplified passive modes. A higher coupling parameter K is obtained near the cutoff for delay-lines with a filter character. The calculation yields $G/N=37.5\left(\frac{\omega_p\infty}{\omega}\right)^{2/3}$ (decibel) for the maximum attainable amplification per beam wave-length of an idealized T.W.T. amplifier. This value, it will be seen, depends, as in all space-charge wave-amplifiers, on the ratio:

$$\frac{\omega_p\infty}{\omega}=\frac{\text{plasma frequency for an infinite electron cloud,}}{\text{circuit frequency}}$$

i.e., the gain depends on the current density and not on the current of the beam. *H. W. A. Gerlach.*

Durandeau, Pierre.; et Fert, Charles. Lentilles électroniques magnétiques. Rev. Opt. **36** (1957), 205–234.

A collection and description of the theoretical considerations involved in the practical construction of electromagnetic lenses. The conditions required in making the magnetic circuit depend on the uses for which the lens is to be designed. Rules are given for the polar faces so that the field shall have a minimal spread. The electro-optical properties are expressed as universal functions of the construction data. *E. W. Marchand.*

Kohler, Max. Invariante Flächen der Elektrodynamik. Z. Physik **148** (1957), 443–453.

Aus dem schiefsymmetrischen Tensor M_{ik} des elektromagnetischen Feldes geht durch den bekannten Stokesschen Prozeß der Tensor

$$\frac{\partial M_{ik}}{\partial x_l}+\frac{\partial M_{li}}{\partial x_k}+\frac{\partial M_{kl}}{\partial x_i}$$

hervor. Sein identisches Verschwinden (im System der Maxwellschen Gleichungen) garantiert bis auf eine Eichtransformation ($\Phi_k'=\Phi_k+\partial\varphi/\partial x_k$) die Existenz des Vierpotentials Φ_i. Die Pfaffsche Form $\omega=\Phi_k dx^k$ ist dann nur bis auf ein vollständiges Differential festgelegt. Daher kann man sich auf Pfaffsche Formen gerader Klasse s beschränken und erhält für die Form ω die beiden Fallunterscheidungen $s=2$ und $s=4$. Man erhält die kanonischen Formen

$$\omega=\xi_1 d\eta_1\ (s=2),\quad \omega=\xi_1 d\eta_1+\xi_2 d\eta_2\ (s=4).$$

Der Zusammenhang der (von einander unabhängigen) Funktionen ξ_i, η_i mit den Feldgrößen M_{ik} und Φ_i ist durch die nichtlinearen Beziehungen

$$\Phi_i=\xi_i\frac{\partial\eta_1}{\partial x_i},\quad M_{ik}=\frac{\partial\xi_1}{\partial x_i}\frac{\partial\eta_1}{\partial x_k}-\frac{\partial\xi_1}{\partial x_k}\frac{\partial\eta_1}{\partial x_i}\quad (s=2),$$

$$\Phi_i=\xi_1\frac{\partial\eta_1}{\partial x_i}+\xi_2\frac{\partial\eta_2}{\partial x_i},$$

$$M_{ik}=\frac{\partial\xi_1}{\partial x_i}\frac{\partial\eta_1}{\partial x_k}-\frac{\partial\xi_1}{\partial x_k}\frac{\partial\eta_1}{\partial x_i}+\frac{\partial\xi_2}{\partial x_i}\frac{\partial\eta_2}{\partial x_k}-\frac{\partial\xi_2}{\partial x_k}\frac{\partial\eta_2}{\partial x_i}\quad (s=4)$$

gegeben. Mit dem System der Skalare ξ_α und η_α ist auch jedes andere System ξ'_α und η_α' zur Darstellung der Φ_i und M_{ik} gleichberechtigt, wofern die Invarianzbedingung

$$\sum_{\alpha=1}^{r} \xi_\alpha d\eta_\alpha = \sum_{\alpha=1}^{r} \xi_\alpha' d\eta_\alpha'$$

erfüllt wird. Im Falle $s=2$ genügen die Komponenten M_{ik} der für Plückersche Tensoren charakteristischen Beziehung

$$M_{12}M_{34}+M_{13}M_{42}+M_{14}M_{23}=0.$$

Als physikalische Deutung ergibt sich in diesem Falle die Orthogonalität $\mathfrak{b}\mathfrak{B}=0$ des elektrischen und magnetischen Vektors in jedem Raumpunkt und zu jeder Zeit. Sollen die Linienelemente der Orthogonaltrajektorien dx_k einer durch M_{ik} gegebenen Flächenrichtung sich auch zu 2-dimensionalen Flächen vereinigen, so muß das Pfaffsche System $M_{ik}dx^k=0$ $(i=1,2,3,4)$ vollständig integrabel sein. Dies gilt im Falle $s=2$ stets und führt auf die zweiparametrige Schar zweidimensionaler Fläche $\xi_1=\text{const}$ and $\eta_1=\text{const}$. Diese gegenüber Koordinatentransformationen invarianten Flächen sind sowohl von metrischen Voraussetzungen frei (wie auch der Stokes'sche Prozeß) als auch von solchen über die Viererstromdichte P^i (in der zweiten Gruppe der Maxwellschen Gleichhungen). Setzt man insbesondere $P^i=0$ und verwendet die Metrik der speziellen Relativitätstheorie, so ergibt sich eine zweite invariant definierte zweiparametrige Flächenschar ($f_1=\text{const}$ und $f_2=\text{const}$). Man erhält so im allgemeinen durch jeden Weltpunkt ein Paar orthogonaler Flächen aus der einen und der anderen Schar zu jeder Lösung $(\mathfrak{b}, \mathfrak{B})$ der strom- und ladungsfreien Maxwellschen Gleichungen mit $\mathfrak{b}\mathfrak{B}=0$. Sind die f_1, f_2 von den ξ_1, η_1 unabhängig, so schneiden die beiden Weltflächen einander nur in einem gemeinsamen Weltpunkt, andernfalls in einer geodätischen Isotropen. Als spezielle Beispiele behandelt Verfasser noch das elektrostatische Feld, das stationäre Magnetfeld und den Fall einer ebenen Welle. *M. Pinl* (Köln).

McCrea, W. H. Hertzian electromagnetic potentials. Proc. Roy. Soc. London. Ser. A. **240** (1957), 447–457.

Formulae involving Hertzian electromagnetic potentials, developed in 3-vector notation by A. Nisbet [same Proc. **231** (1955), 250–263; MR **18**, 700], are here rederived in tensor form for any curvilinear coordinates in flat space-time. The basic equations are the Maxwell equations

$$(1) \qquad (H^{pq})_q=J^p, \quad (^*F^{pq})_q=0,$$

and the constitutive equation

$$(2) \qquad H_{pq}+P_{pq}=F_{pq},$$

where the subscripts in (1) indicate covariant derivates and * indicates the dual. The main result is that (1) and (2) are satisfied by

$$H^{pq}=\eta^{pqrs}(^*W_s{}^t)_{tr}+Q^{pq}, \quad ^*F^{pq}=\eta^{pqrs}(W_s{}^t)_{tr}+{}^*R^{pq},$$

where η^{pqrs} is the Levi-Civita tensor, $(Q^{pq}, {}^*R^{pq})$ are any particular solutions of (1), and W_{pq} (the Hertzian potential) is any skew-symmetric tensor satisfying

$$\Box^2 W_{pq}+P_{pq}+Q_{pq}-R_{pq}=0,$$

$\Box^2$ being the d'Alembertian. Attention is paid to gauge transformations of the vector potential $A_p=(W_p{}^q)_q$ and the tensors Q^{pq}, R^{pq}, W^{pq}. For the case where $J_p=0$, $P_{pq}=0$, it is shown that W_{pq} may be reduced to canonical form, a result connected with a theorem of E. T. Whittaker. {The reviewer is not clear as to the physical situation to which the formulae (1) and (2) apply. Nisbet was concerned solely with a medium at rest, but the space-time formulation suggests application to a medium in uniform motion. However the usual constitutive equations for a moving medium involve its 4-velocity, and are very different from (2); cf. C. Møller, The theory of relativity, p. 201 [Oxford, 1952; MR **14**, 212] and J. L. Synge, Relativity: the special theory, p. 416 [Interscience, New Yrok, 1956; MR **17**, 1013].} *J. L. Synge* (Dublin).

Wild, E. Electromagnetic waves in nearly periodic structures. Quart. J. Mech. Appl. Math. **10** (1957), 322–341.

The theory of electromagnetic waves in a nearly periodic cavity like those of certain linear accelerators is treated by a perturbation of the solution for a periodic structure. All apertures connecting successive structural elements must be congruent for the theory to apply. An assumption of completeness for the characteristic aperture fields is also required. The results are applied to a structure of the type of the Berkeley proton accelerator. *A. A. Blank* (New York, N.Y.).

Kline, Morris. Electromagnetic research at the Institute of Mathematical Sciences of New York University. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. EM–93 (1956), 19 pp.

A useful summary of the investigations of the Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, 1949–1956, covering mathematical developments and specific results in diffraction theory and wave propagation. Most of this research has already been reported upon in these reviews. *A. A. Blank* (New York, N.Y.).

Carini, Giovanni. Su una soluzione rigorosa delle equazioni di Minkowski per i conduttori in moto e sua interpretazione fisica. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **2** (1955–56), 253–260.

Carini, Giovanni. Sull'equazione dell'impulso elettromagnetico nell'elettrodinamica dei corpi in moto. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **2** (1955–56), 283–291.

Crupi, Giovanni. Sulle onde elettromagnetiche nei conduttori in moto. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **2** (1955–56), 299–307.

Crupi, Giovanni. Sulle onde elettromagnetiche di forma impulsiva nei corpi in moto. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **3** (1956–57), 73–87.

Totaro, Carmelo. Sull'equazione delle onde nei corpi in moto di G. Lampariello. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **2** (1955–56), 293–298.

Totaro, Carmelo. Considerazioni sui fenomeni di riflessione e di rifrazione nei corpi in moto. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **3** (1956–57), 89–94.

These six papers all deal with electromagnetism in a medium S_0 which moves with uniform velocity relative to the frame S of some Galileian observer. All the authors work with the 3-vectors (**E**, **H**, **D**, **B**, **I**) in the frame S, instead of using 4-dimensional notation. Use is made of the equation

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\Delta\Phi+\frac{n^2-1}{c^2-v^2}\frac{d^2\Phi}{dt^2}+\frac{\sigma\mu}{c(c^2-v^2)^{\frac{1}{2}}}\frac{d\Phi}{dt}=0,$$

due to G. Lampariello [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 37–44, 222–228; MR **16**, 775, 1337, **17**, 216] and G. Carini [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 41–47; MR **17**, 560]. This equation is satisfied by each of the components of the electromagnetic vectors; in it $n^2=\varepsilon\mu$, v=speed of S_0, and d/dt means differentiation following S_0. It appears to the reviewer that, while this equation is valid in the case of a dielectric ($\sigma=0$), it is true for a conductor ($\sigma\neq 0$) only when the field is such that the charge-density is zero in the frame S_0, a condition satisfied for all time if satisfied for $t=0$. The authors do not mention this restriction, but they write the constitutive equations involving current and charge-density in a form which is valid only under this restriction. Since the Maxwell-Minkowski equations are Lorentz-invariant (a fact at once apparent when they are written in 4-dimensional notation), the results obtained in these papers are not very exciting, because they might all have been obtained in a routine way by applying a Lorentz transformation to well-known results for a fixed medium.

In his first paper, Carini studies waves in which the vectors are of the form $\mathbf{E}=\mathbf{K}\exp i(k\mathbf{u}\cdot\mathbf{r}-\omega t)$, and, in his second paper discusses the justification of the Minkowski energy-tensor [a matter much easier to treat, in the opinion of the reviewer, by the standard 4-dimensional methods]. In his first paper, Crupi also studies simple waves as above; in his second paper he deals with electromagnetic shock waves. In his first paper, Totaro obtains, for Lampariello's equation ($\sigma=0$), the most general solution depending only on (x, t); in his second paper he studies reflection and refraction at the surface of separation of two dielectrics moving with the same velocity.

J. L. Synge (Dublin).

Toraldo di Francia, Giuliano. Onde elettromagnetiche piane in un mezzo dielettrico non omogeneo, corrispondente a uno spazio di Fermat pseudosferico. Boll. Un. Mat. Ital. (3) **11** (1956), 515–523.

Verf. betrachtet ein homogenes, isotropes Medium mit konstanter, magnetischer Permeabilität und mit einer Dielektrizitätskonstanten, die proportional zum reziproken Quadrat des Abstandes von einer Ebene ist. Er geht von den Maxwellschen Gleichungen in einem solchen Medium aus und separiert die betreffenden Variablen. Er zeigt dann, wie eine ebene TEM-Welle sich im Medium fortpflanzt. In bezug auf Anwendungen auf Wellenleiter betrachtet Verf. TE-Wellen, welche sich in gekrümmten Wellenleitern fortpflanzen. Er zeigt, dass eine entsprechence örtliche Aenderung der Dielektrizitätskonstanten derart geschaffen werden kann, dass sich eine Welle mit ebener Front fortpflanzt. *M. J. O. Strutt.*

Grinberg, G. G.; Lebedev, N. N.; Skal'skaia, I. P.; and Ufliand, Ia. S. The electromagnetic field of a linear emitter located inside an ideally conducting parabolic screen. Soviet Physics. JETP **3** (1956), 366–378.

A linear emitter, current strength $I_0\exp(i\omega t)$, lies along the focal line of a parabolic cylinder. The radiation field is derived with detailed mathematical discussion, including the limiting case of geometrical optics. A somewhat more general location of the emitter is also considered. *C. Strachan* (Aberdeen).

Kirkham, Don. Potential and capacity of concentric coaxial capped cylinders. J. Appl. Phys. **28** (1957), 724–731.

The author has computed the capacity of a finite length of conducting cylinder placed symmetrically in a cylindrical cavity. The method consists of breaking the inseparable region into two separable regions and matching the solutions on the common boundary. This part of the technique is identical to that used by W. C. Hahn [same J. **12** (1941), 62–68; MR **2**, 334; see also N. A. Begovich, Hughes Aircraft **TM–234 A**] but in the present instance the author approximates the infinite set of equations by a finite set, and does not attempt to find a dominant series.

W. K. Saunders (Washington, D.C.).

Brahmachary, R. L. A generalization of Reissner-Nordström solution. II. Nuovo Cimento (10) **5** (1957), 1520–1523.

This is a continuation of the work on a generalization of Reissner-Nordström solution, namely the interior solution of a charged (material) sphere (MR **18**, 543]. Following Tolman [Phys. Rev. (2) **55** (1939), 364–373] we may easily obtain two special solutions of the field-equations, when the energy-tensor consists of two parts, namely the electromagnetic and the material parts. The second solution, obtained by assuming $e^{\nu}(\nu'/2r)$=const is discussed here. We further obtain another solution based on the very restrictive assumption $\rho=-p$. We also show the validity of Birkhoff's theorem for the case of a charged sphere. *Author's summary.*

Boillet, Pierre. Sur l'interprétation du principe de Huygens: cas des ondes acoustiques, élastiques et électromagnétiques. Cahiers de Phys. no. **78** (1957), 59–87.

This is the first part of a Paris doctorate thesis. The author's declared objective is to try to show that the intuitive idea which Huygens had in formulating his principle is justifiable on the basis of the mathematical theory of wave propagation and that the idea of secondary sources can be given a precise physically realizable definition. He carries out his program for the case of sound waves of small amplitude, and reaches the conclusion that Kirchhoff's formula provides a proper mathematical formulation of Huygen's Principle.

Elastic and electromagnetic wave motion will be discussed in a continuation of the paper. *E. T. Copson.*

★**Franz, W. Theorie der Beugung elektromagnetischer Wellen.** Ergebnisse der angewandten Mathematik. Bd. 4. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. iv+123 pp.

The task of writing a comprehensive account of modern diffraction theory would lead a writer in this field to a book of treatise length. Aside from all of the interesting mathematical developments in this field in the past two decades, there are numerous specific problems which should be discussed. Accordingly, the present author limits himself in this monograph to three aspects of this subject. This text will serve to provide the reader with an indication of most of the classical aspects of this subject.

The three major topics discussed are (a) the representation theorems, (b) diffraction of an object without edges and (c) diffraction of an object with edges. The first topic discusses a method of formulating problems in the electromagnetic theory of diffraction — that is a vector Green's function formulation. To this end, the dyadic Green's function is introduced and integral equations describing some of the boundary value problems are discussed. Here one can object to the use of the dyadic Green's function which obeys a second order equation, while the Maxwell equations are of the first order. A

number of mathematical difficulties arise which can be avoided by a first order representation, that is the scalar Green's function and its derivatives [see C. Müller, Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen, Springer, Berlin, 1957, Ch. VI, VII; MR **19**, 209].

Of the two remaining topics, (b) deals with diffraction of waves by such bodies as spheres or in general "sufficiently smooth surfaces". While there are a number of problems which may be done explicitly in this case, a good deal of analysis is required in order to discuss the nature of the solution for large wave numbers — a topic to which the author gives much attention.

The third topic is concerned with such domains as the wedge and half-plane, and the method of Sommerfeld for the solution of such problems is discussed. Various approximation methods for such problems, including Kirchoff's, are considered. An indicative bibliography is appended. *A. E. Heins* (Pittsburgh, Pa.).

★ **Fock, V. A. Diffraction, refraction, and reflection of radio waves.** With an introduction by V. I. Smirnov and an appendix by M. A. Leontovich. Antenna Laboratory, Air Force Cambridge Research Center, Bedford, Mass., June 1957. v+xx+391 pp.

This volume contains English versions of the following papers by V. A. Fock: 1. New methods in diffraction theory. MR **9**, 637. 2. The distribution of current induced by a plane wave on the surface of a conductor. MR **8**, 185. 3. Diffraction of radio waves around the earth's surface. MR **17**, 559. 4. Solution of the problem of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation. Ž. Eksper. Teoret. Fiz. **10** (1946), 13–24. 5. The field of a plane wave near the surface of a conducting body. MR **8**, 363. 6. Propagation of the direct wave around the earth with due account for diffraction and refraction. MR **10**, 89. 7. Theory of radiowave propagation in an inhomogeneous atmosphere for a raised source. MR **17**, 559. 8. The field from a vertical and a horizontal dipole, raised above the earth's surface. MR **11**, 863. 9. Fresnel diffraction from convex bodies. MR **13**, 514. 10. Fresnel reflection laws and diffraction laws. Uspehi Fiz. Nauk **36** (1948), 308–319. 11. Generalization of the reflection formulas to the case of reflection of an arbitrary wave from a surface of arbitrary form. MR **12**, 655. 12. Approximate formula for distance of the horizon in the presence of superrefraction. Radiotehn. i Elektr. **1** (1956) 560–574. 13. On radiowave propagation near the horizon with superrefraction. Radiotehn. i Elektr. **1** (1956) 575–592.

★ **Justice, Raymond. Scattering by a small conducting prolate spheroid.** Contract No. AF 19(604)–1725. AFCRC–TN–56–750–AD 98804–ASTIA Document No.: AD 9880–4. The Ohio State University Research Foundation, Columbus, Ohio, 1956. iv+79 pp.

In this report the author has investigated the problem of diffraction of a plane electromagnetic wave by a perfectly conducting prolate spheroid. The solution of this problem is based on the assumption that the electromagnetic fields E and H (incident and diffracted) can be expanded in ascending powers of $k=2\pi/\lambda$. From Maxwell equations and the boundary conditions a set of first order non-homogeneous equations are obtained for the coefficients of the series, when one equates terms of like powers in k. The leading set represent the static fields, whereas the higher order coefficients (fields) satisfy homogeneous vector differential equations of the second, third, etc. orders; The first order diffracted fields E_1^d, H_1^d satisfy the equations, curl curl E_1^d=curl curl H_1^d=0. These fields are found by first solving the Laplace equation in spheroidal coordinates for the auxiliary functions ϕ_1 and ψ_1 respectively, and then taking the appropriate Stratton-Chu combinations. However, the higher order fields cannot be evaluated in this way as the higher order vector equations are not tractable by present methods. The author has employed two distinct methods for obtaining the first and second order diffracted fields. In the first method (for normal incidence only) the vector mode functions are used which are found by taking the proper Stratton-Chu combinations with arbitrary coefficients. The latter are determined from the boundary conditions and the radiation condition. In the second method, which is based on potential theory, the author treats the more general case of oblique incidence. A comparison of the two methods for the fields up to order k^{-2} shows that they are equivalent.

It should be pointed out that the results presented here depend on the existence of an ascending power series solution in k for the incident and diffracted fields and that the individual terms (fields) satisfy the boundary and radiation conditions. The first assumption is satisfied, provided that the scattering obstacle is smooth (continuous turning tangent plane) and its dimensions are small. However, such series in general are slowly convergent and depend on positive powers of the distance. Likewise, the assumption that the individual terms of the series satisfy the boundary and radiation condition does not hold, since the functions employed here for calculating the fields are not the appropriate eigenfunctions. The author has realized these difficulties in carrying out his calculations, by limiting his treatment to the second order fields. [The problem of diffraction of plane electromagnetic waves incident in the direction of the long axis of the prolate spheroid has been treated by several authors: see Siegel, Schultz, Gere, and Sleator, I.R.E. Trans. **AP–4** (1956), 266–275, where full references are given.] *N. Chako* (Flushing, N.Y.).

Jauch, J. M. On the relation between scattering phase and bound states. Helv. Phys. Acta **30** (1957), 143–156.

From the author's abstract: "The relation between phase shifts and bound states proved by Levinson for spherical symmetric potential functions satisfying certain regularity conditions [Danske Vid. Selsk. Mat.-Fys. Medd. **25** (1949), no. 9; MR **10**, 710] is generalized for arbitrary interactions. The regularity condition of Levinson is replaced by a condition for the behavior of the scattering state wave functions at infinite energy. The proof of the relation given here shows that it is a simple consequence of the orthogonality and completeness relation for the eigenfunctions of the total energy operator." *N. Levinson* (Cambridge, Mass.).

Berceli, T. Surface wave propagation along coated wires. Acta Tech. Acad. Sci. Hungar. **17** (1957), 219–252. (German, French and Russian summaries)

The author considers a single-wire transmission line the coating of which is both dielectric and magnetic. He discusses for the case of small radii the phase and group velocities, phase distortion, and power losses. He shows the existence of an optimum for power transmission and presents a procedure for the practical design of such lines. *C. H. Papas* (Pasadena, Calif.).

★**Toraldo di Francia, G. Un problema sulle geodetiche delle superficie di rotazione che si presenta nella tecnica delle microonde.** Estratto dagli "Atti della fondazione Giorgio Ronchi", anno XII, N. 2, Marzo-Aprile 1957, pp. 151–172. Tipografia Giuseppe Bruschi, Firenze.

The problem of determining a surface of rotation such that any geodesic leaving a given parallel under an angle ψ in a point M, intersects again the same parallel at a longitude $\varphi=F(\psi)$ from M, is treated. The solution depends upon an Abel integral equation. An application is made to the case of Rinehart-Luneberg surfaces.

The results are discussed as far as their application to the design of scanning devices in microwave techniques, from which the research originates, are concerned.

A. Marussi (Trieste).

Kondo, Kazuo; and Uehara, Takeyuki. A preliminary note on a stochastical interpretation of dynamical equations. Matrix Tensor Quart. **7** (1957), 77–83.

In the first part of the paper the author argues that in rheonomic dynamical systems (such as a number of charged particles in an external electromagnetic field) the rheonomic terms may be attributed to hidden degrees of freedom lying outside the field of observation. Extending this viewpoint it is shown that in averaging a large number of fluctuating quantities the mean is also a rheonomic variable. In the second part of the paper the equations of motion of such dynamical systems are established. The statistical procedure is shown to be equivalent to the introduction of a number of non-holonomic constraints, leading to the Boltzmann-Hamel equation. Because of the special mathematical form of these constraints, the non-holonomic object becomes a torsion tensor and the non-holonomic Riemannian space becomes holonomic, but non-Riemannian. The averaging of fluctuating variables is analogous to the commutation of alternating-currents in the large number of conductors of a generator into an average direct-current. The dynamical equations of a rheonomic system with stochastic variables become thus analogous to those of rotating electrical machinery. *G. Kron* (Schenectady, N.Y.).

Schäffer, J. J. Zur Theorie der elektrischen Netzwerke mit nichtlinearen Elementen. Arch. Elektrotech. **43** (1957), 151–168.

A general study of the forced oscillation of non-linear electrical RC and RL networks with arbitrary number of meshes is undertaken. The resistances are linear, the capacitors (or inductors) are non-linear (hysteresis excluded) and the forcing functions (impressed voltages or currents only) are almost-periodic. It is shown that for arbitrary initial conditions the network responses converge asymptotically to a single almost-periodic solution, the stationary solution. Each frequency in the spectrum of the stationary answer is a linear combination with integer coefficients of the frequencies in the forcing functions. Under special conditions the results are valid for RLC networks also, as demonstrated by the detailed solution of a five-mesh network. *G. Kron* (Schenectady, N.Y.).

Kucera, Jaroslav. La théorie du transformateur dans l'analyse tensorielle. Rev. Gén. Elec. **66** (1957), 279–291.

The interconnection and phase reduction of multi-winding transformers is illustrated by the method of tensors. *G. Kron* (Schenectady, N.Y.).

★**Bose, Amar G. A theory of nonlinear systems.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. 309 (1956). i+58 pp.

Author's Abstract: "The Wiener theory of nonlinear system characterization is described and some of its important concepts are discussed. Following these lines, a theory is developed for the experimental determination of optimum time-invariant nonlinear systems. The systems are optimum in a weighted mean-square sense in which the weighing function is at our disposal.

"The design of nonlinear systems is regarded as the problem of mapping the function space of the past of the input onto a line that corresponds to the amplitude of the filter output. By choosing a series expansion for this mapping operation that partitions the function space into nonoverlapping cells, an orthogonal representation for nonlinear systems is obtained that leads to convenient apparatus for the determination of optimum systems. General methods are described for applying this theory to the determination of systems that have a performance superior to that of given linear or nonlinear systems. A criterion is established relative to which two systems are defined as "nearly equivalent" and the approximation of nonlinear systems by linear and simple nonlinear systems is discussed. The theory is extended to include the problem of multiple nonlinear prediction and apparatus for the determination of optimum predictors is indicated."

N. Levinson (Cambridge, Mass.).

de Buhr, Johann. Die geometrische Darstellungsweise hintereinandergeschalteter, allgemeiner, verlustbehafteter Vierpole. Arch. Elek. Übertr. **11** (1957), 173–176.

The cascade connection of several general dissipative 4-poles can be represented in a purely geometrical way. This representation allows one to obtain the resulting 4-pole equivalent to the cascade system. Pure reactance 4-poles can be represented in the plane of the unit circle; dissipative 4-poles must be represented in threedimensional space. Presentation in the drawing plane is made possible by stereographic projection. (From the author's summary.) *B. Gross* (Rio de Janeiro).

★**Blokh, A. Sh. Synthesis of (p, q)-terminal switching networks.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 1017–1019.

See also: Ordinary Differential Equations: Babister. **Partial Differential Equations:** Karp. **Numerical Methods:** Rybner. **Physical Applications:** Finzi. **Fluid Mechanics, Acoustics:** Hochstadt; Abbi and Chandra. **Relativity:** Hessaby; Trautman.

Classical Thermodynamics, Heat Transfer

Péneloux, André. Démonstration des relations réciproques de Gibert pour les systèmes chimiques. C. R. Acad. Sci. Paris **244** (1957), 2238–2241.

The paper gives an alternative, more rigorous proof of the reciprocal relation due to R. Gibert [same C. R. **236** (1953), 2145–2147] for a chemical system in which two independent chemical reactions take place simultaneously. The argument is macroscopic in character. If v_1 and v_2

denote the reaction rates and A_1 and A_2 denote the affinities (as defined by Defay), then

$$\frac{\partial v_1}{\partial A_2}=\frac{\partial v_2}{\partial A_1}.$$

J. Kestin (Providence, R.I.).

Kasahara, Eiji. Conduction of heat in a circular cylinder due to the source in form of a circular ring moving axially on the surface of the cylinder with constant speed. Proc. Fac. Engrg. Keio Univ. **9** (1956), 13–20 (1957).

By use of Green's function the author obtains both steady and non-steady solutions of the title problem. They give the temperature distribution and basic relations for calculation of the thermal stresses and strains in the cylinder. *From the author's summary.*

★ **Borden, Avis. Time constants and frequency response of coated hot wires used as turbulence-sensing elements.** Hydromechanics Laboratory, Report 952. David Taylor Model Basin, June, 1957. viii+48 pp.

The problem of non-steady heat flow through a cylindrical wire surrounded by a thin cylindrical shell is solved by Laplace transforms for the following cases: a constant-current coated wire in which the rate of convective cooling changes; a constant-current coated wire in which the current input changes; a constant-temperature wire in which the rate of convective cooling changes. Initiating disturbances are at first assumed to be step functions; then in each case the response to a periodic function is obtained by Duhamel's theorem. The mathematical evolvement is weakened by the assumption, without argument, of certain "identities" in the coefficients appearing in the solutions. *R. N. Goss* (San Diego, Calif.).

★ **Markstein, George H. Analysis of steady, finite-amplitude cellular flames.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 295–320. Stanford University Press, Stanford, Calif. $7.50.

An analysis is made for the case of steady, two-dimensional cellular flames of finite amplitude, assuming incompressible and inviscid flow. The flame front is treated as a surface of discontinuity separating gases of constant densities, the burning velocity being introduced as a function of the local curvature of the flame front. Use is made of a perturbation method analogous to that used in analyzing finite amplitude gravity waves. Good agreement with experiment is obtained in predicting the existence of a minimum cell width at which sudden collapse of flame cells occurs, and of a maximum width for cell splitting. Computed flame profiles also gave good agreement with experiment. *J. F. Heyda* (Cincinnati, Ohio).

★ **Dooley, Donald A. Ignition in the laminar boundary layer of a heated plate.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 321–342. Stanford University Press, Stanford, Calif. $7.50.

Analysis considers ignition and combustion in the laminar boundary layer of a constant temperature, semi-infinite flat plate. A one-step unopposed "global" reaction following any order reaction kinetics with temperature dependence according to the Arrhenius rate law is assumed. For the case where the Prandtl and Schmidt numbers are equal, the determination of a similarity function relating the species concentrations to the local temperature greatly simplifies the analysis. The similarity function is shown to be equal to the dimensionless streamwise velocity when the Prandtl and Schmidt numbers are both equal to unity. A general analytic solution for the nth approximation to the temperature and concentration profiles in the reacting laminar boundary layer is obtained. *From the author's summary.*

See also: Numerical Methods: Manfredi. **Fluid Mechanics, Acoustics:** Rubesin and Inouye.

Quantum Mechanics

Lochak, Georges. A propos des fluides classiques à spin. C. R. Acad. Sci. Paris **244** (1957), 2291–2293.

A variational principle is given from which the equations of Weyssenhoff for a classical fluid with spin may be derived. *A. H. Taub* (Urbana, Ill.).

Feyerabend, P. K. Zur Quantentheorie der Messung. Z. Physik **148** (1957), 551–559.

Nach Dirac heißt eine dynamische Variable R, deren Eigenzustände ein vollständiges System bilden eine Observable. Die Eigenzustände sind dann durch das System der zu R gehörigen Eigenfunktionen $|\varphi_1\rangle$, $|\varphi_2\rangle$, $\cdots$ und Eigenwerte λ_1, λ_2, $\cdots$ gegeben. (Verfasser benützt die Diracsche Schreibweise für "Bra and Ket vectors".) Der Einfachheit halber werden nur zwei Eigenzustände zu Grunde gelegt $|\varphi_1\rangle$ und $|\varphi_2\rangle$ und zunächst

$$(*) \qquad |\langle\Phi|\varphi_1\rangle|^2\neq|\langle\Phi|\varphi_2\rangle|^2$$

angenommen. Dabei bedeutet $|\Phi\rangle$ den Anfangszustand des Systems S, an welchem R gemessen werden soll. Ferner sollen die Zustände des Meßapparates M, die $|\varphi_1\rangle$ und $|\varphi_2\rangle$ entsprechen, d.h. die Eigenzustände $|\psi_1\rangle$ und $|\psi_2\rangle$ der "Apparatevariablen" R' makroskopisch unterscheidbar sein. (M enthält etwa einen Zeiger, der nur zweier Positionen fähig ist, A und B, und man kann durch bloßes Hinsehen feststellen, ob A vorliegt oder ob B vorliegt). Verfasser teilt den Meßprozeß in folgende drei Stadien ein: (1) Wechselwirkung zwischen S und M, bis der Zustand des Gesamtsystems $\{S+M\}$ die Form

$$\exp[-i/\hbar\cdot HT]|\Phi\Psi\rangle=|\Phi\Psi\rangle'=\sum_i\langle\Phi|\varphi_i\rangle|\varphi_i\rangle|\psi_i\rangle,$$

(T=Zeit der Wechselwirkung)

hat (H Hamiltonoperator des Gesamtsystems, $|\Psi\rangle$ Anfangszustand des Meßapparates). (2) die unvollständige Messung von R' (M wird z.B. photographiert oder einfach angeschaut) und die direkte Darstellung dieser unvollständigen Messung. (3) Vervollständigung der Messung von R' mit dem Ergebnis: S und M befinden sich in einem bestimmten und bekannten Zustand. — Auf Grund der klassischen Auffassung befindet sich S und M bereits am End des Stadiums (1) in einem wohlbestimmten Zustand (d.h. Schrödingers Katze im Kasten mit Uranatom und Zählrohr, das beim Ansprechen einen Hebel auslöst, der einen Behälter mit Cyanwasserstoff zerschlägt, ist nach einem Tag entweder tot oder lebendig), der unverändert bleibt, ob nun ein Beobachter seine Aufmerksamkeit dem Meßinstrument zuwendet oder nicht. Nach der Quantenmechanik ist der Zustand von $\{S+M\}$ am Ende der Wechselwirkung durch den statistischen Operator P' charakterisiert, während sich $\{S+M\}$ im Zustand des

Gemisches P_M befindet, wenn man annimmt, der Zustand von M sei zwar bereits fixiert, aber noch nicht bekannt. Da $P' \neq P_M$ gilt, kann man am Ende des Stadiums (1) noch nicht sagen, daß sich M in einem wohldefinierten, wenn auch noch unbekannten Zustand befindet (Schrödinger's Katze ist weder lebendig noch tot, sondern erst nach der Beobachtung wirklich entweder lebendig oder tot). Damit wird das Stadium (2), d.h. der Vorgang $P' \to P_M$ ein notwendiges Element des Meßprozesses. Selbst wenn man annimmt, daß man schon am Ende des Stadiums (2) den Zustand von $\{S+M\}$ durch P_M kennzeichnen dürfte, so ergibt eine neue Schwierigkeit, wenn (*) nicht besteht. Eine dritte Schwierigkeit ergibt sich durch den Umstand, daß jede Messung zu irreversiblen Veränderungen führt. So kommt Verfasser zur Vermutung, daß die Heisenberg-von Neumannsche Theorie des Meßprozesses zwar korrekt jedoch unvollständig ist. Eine vollständige Theorie muß auch dem makroskopischen Charakter des Meßinstruments und des Ablesungsprozesses Rechnung tragen. Dazu wird das folgende "Beobachterprinzip" eingeführt: wenn die Observable R für B beobachtbar ist, dann ist S für B dann und nur dann beobachtbar, wenn S mit R kommutiert. Daraus folgt: für einen Makrobeobachter ist P' mit P_M identisch. Ferner widerspricht das Beobachterprinzip nicht dem "Diagonalprinzip": makroskopische Eigenschaften sind immer diagonal. *M. Pinl* (Köln).

Kato, Tosio. On the eigenfunctions of many-particle systems in quantum mechanics. Comm. Pure Appl. Math. **10** (1957), 151–177.

The Hamiltonian

$$H=-\sum_{i=1}^{s}\mu_i \operatorname{grad}_i^2-\mu_0\Big(\sum_{i=1}^{s}\operatorname{grad}_i\Big)^2+W(\mathfrak{r}_1,\cdots,\mathfrak{r}_s),$$

where $\mathfrak{r}_i=(x_i, y_i, z_i)$ is the three-dimensional position vector of the 'ith particle', while the μ_i are non-negative constants (strictly positive for $i>0$) was shown by the author to be essentially self-adjoint on a natural domain in the space of square-integrable functions of $3s$ variables ($i=1, 2, \cdots, s$), provided

$$W(r_1,\cdots,r_s)=W_0(r_1,\cdots,r_s)+\sum_{i=1}^{s}V_{0i}(r_i)+\sum_{i<j}V_{ij}(r_i-r_j),$$

where W_0 is real-valued, bounded, and measurable; for each i and j the V_{ij} are real-valued measurable functions vanishing outside some sphere and satisfying the inequality $\int|V_{ij}(\mathfrak{r})|^{\sigma}d\mathfrak{r}<\infty$, where σ is a constant $\geqq 2$. The eigen-functions of the present article are defined as functions in the domain of all positive integral powers of the self-adjoint extension of H, and so always span a dense subspace, and include proper eigenfunctions in the conventional sense. In the case of the helium atom the singularities of the latter had been treated by a number of authors, but with rather inconclusive results. In the present paper it is shown by a novel method that even the generalized eigenfunctions are better behaved than had been expected.

Specifically it is shown that all eigenfunctions are bounded and are C_θ with any $\theta<2-3/\sigma$ and <1, where the class C_θ consists of all functions f on $3s$-dimensional space such that $|f(P)-f(Q)|\leq C\overline{PQ}^{\theta}$, where P and Q are any points in the space and $\overline{PQ}$ denotes the $3s$-dimensional distance. Moreover, if $\sigma>3$, the first-order partial derivatives are in C_θ. A generalized Coulomb potential is defined as one in which $V_{ij}(r)=e_{ij}\eta(r)r^{-1}+V_{ij}'(r)$, where $r=r$, the e_{ij} are constants, $\eta(r)$ is the characteristic function of an interval $[0, \rho]$, $\rho>0$, and the V_{ij}' satisfy the same conditions as the V_{ij} above with $\sigma>3$. In this case it is shown that all eigenfunctions are continuous and have derivatives of first order except at the points where one of the r_i or r_{ij} vanishes. Rather precise information concerning the behavior of the eigenfunction at the singularities (analogous to that for the hydrogen atom wave functions) is also obtained. The proofs are based on functional analysis and are of notable lucidity and rigor. *I. E. Segal* (Chicago, Ill.).

Tzou, Kuo-Hsien. Comparaison détaillée des champs tensoriels aux champs de spin maximum entier en ce qui concerne leur composition. C. R. Acad. Sci. Paris **244** (1957), 2009–2011.

Author's Summary: An analysis based on the Clebsch-Gordon theorem has shown that, with respect to states of spin, a tensor field of rank n has same composition as the field of spin maximum n of the fusion theory [Tzou, same C. R. **244** (1957), 1886–1888; MR **19**, 220]. By adding some precision in the application of the theorem one may also compare in detail the rank, the parity, and the energy sign of component fields. *M. Loève.*

Demkov, Yu. N. Causality in quantum mechanics. Vestnik Leningrad. Univ. **11** (1956), no. 22, 5–11. (Russian)

The role of causality is discussed in classical physics and in quantum mechanics. It is suggested that there exist "anti-causal" particles for which the effect precedes the cause. These are identified with particles possessing negative energy. It is conceded that there is as yet no experimental evidence for their existence. *N. Rosen.*

Gourdin, Michel; et Martin, André. Détermination d'un potentiel séparable à partir des déphasages. Approximation indépendante de forme. C. R. Acad. Sci. Paris **244** (1957), 1153–1154.

Author's summary: "On cherche a déterminer un potentiel non local séparable fournissant pour le système proton-neutron un déphasage S dont la variation avec l'énergie soit donné exactement par l'approximation indépendante de forme. Dans l'état singulet on obtient une solution unique. Dans l'état triplet une famille de potentiels à un paramètre satisfait les conditions requises."

The phase is assumed to be given by $\tan\delta_s=k(\frac{1}{2}r_0k^2-a^{-1})^{-1}$ which leads to a Fredholm equation which is solved by a quotient of two polynomials. *N. Levinson* (Cambridge, Mass.).

Gourdin, Michel; et Martin, André. Détermination d'un potentiel séparable à partir des déphasages. Théorèmes généraux. C. R. Acad. Sci. Paris **244** (1957), 1469–1470.

The uniqueness of the inverse problem is investigated for the case of non-local separable potentials V defined by $V\psi=\varepsilon u(r)\int u(r')\psi(r')d^3r'$ with ε a real constant. Given the s-phase shift, V is determined by a linear inhomogeneous integral equation. If one assumes that $p^{-1}g(p)=2\pi^{-1}\tan\delta$ vanishes in the limits $p\to 0$ and $p\to\infty$, and that $p^{-1}(\int_0^{\infty}ru(r)\sin prdr)^2$ is continuous and also vanishes in these limits, the following can be proven: (1) If $g(+0)$ and $g(+\infty)$ have the same sign, then $\delta(0)-\delta(\infty)=0$ there is no bound state and the solution is unique, (2) if $g(+0)$ and $g(+\infty)$ have opposite signs, then $g(0)>0$, $\delta(0)-\delta(\infty)=\pi$, there is exactly one bound state, and the solution is not unique. These results are in agreement

with previous work by the same authors [see paper reviewed above] where $g(p)$ was assumed to be a rational function; they also agree with the corresponding case for local potentials. *F. Rohrlich* (Iowa City, Iowa).

Burnat, M. On the solution of a problem involving the Schrödinger equation for infinite three-space. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 224–227. (Russian)

Let q be a real potential in three-space which is continuously differentiable except at a finite number of points $x_1, x_2, \cdots$, where $q=O(|x-x_s|^{-\alpha_s})$ $(x\to x_s, \alpha_s<1.5)$ and is $O(|x|^{-\beta})$ $(\beta<3.5)$ at infinity. Then $B=-\Delta+q$ has a unique self-adjoint realization A in L^2. Let $H(x, y, \lambda^2)=(4\pi)^{-1}|x-y|^{-1}+h(x, y, \lambda^2)$ be the resolvent kernel of A. The author gives the estimates M, $M|y-x_s|^{1-\alpha_s}$, and $M|\log|y-x_s||$ of h according as $\alpha_s<1$, >1 and $=1$. The proof depends on the integral operator method, the integral operator involved satisfying the Fredholm alternative when restricted to suitable Banach spaces. The results are applied to the scattering problem: to solve the equation $Bu=\lambda^2 u$ where $u=\exp i\lambda(\omega, x)+v(x)$ and v satisfies a radiation condition. The author thus obtains a partial generalization of the results of A. Va. Possner [Mat. St. N.S. **32(74)** (1953), 109–156; MR **14**, 755]. *L. Gårding* (Lund).

Skyrme, T. H. R. A variational method in relativistic quantum field theory. Nuovo Cimento (10) **4** (1956), supplemento, 754–757.

It has been shown [Edwards, Nuovo Cimento (10) **4** (1956), supplemento, 711–722; MR **19**, 88] that the propagator of a physical nucleon in the presence of a real meson field can be obtained from the solution of the functional equation

$$(1)\qquad \left(\gamma\frac{\partial}{\partial x_\mu}+M_0+g_0\frac{\delta}{\delta p(x)}\right)W=0,$$

where W is a function of x and a functional of $p(x)$. In order to carry out an approximate integration of (1) a variational principle is proposed which is a generalization to the functional case of the well-known method used in the theory of differential equations.

Application of this method to different types of meson theory has been performed using a "trial functional" similar to the one assumed in the intermediate coupling approximation to non-relativistic field theory. [For more details see Skyrme, Proc. Roy. Soc. London. Ser. A. **231** (1955), 321–335; MR **17**, 220.] *S. Fubini* (Genoa).

Nishijima, Kazuhiko. On the asymptotic conditions in quantum field theory. Progr. Theoret. Phys. **17** (1957), 765–802.

In quantum field theory it is not easy to investigate the asymptotic behavior of a system because the total Hamiltonian cannot readily be divided into a free part and an interaction. This is so because of the selfinteractions. Instead, one has to do the study of the asymptotic behavior using the over-all space-time method. In the present paper the sometimes ambiguous discussions of the limiting procedures are circumvented by the introduction of the asymptotic conditions in a new way. The point is the introduction of recursion formulae for state vectors as the defining equations of a complete set of state vectors. Then the limiting procedures are replaced by algebraic conditions whose selfconsistency can be investigated. In doing this Zimmermann's (unpublished) retarded chronological R product is used, and a set of integral equations is derived for the vacuum expectation values of the R product. These integral equations are similar to those of Chew and Low [Phys. Rev. (2) **101** (1956), 1570–1579] and of Lehmann, Symanzik, and Zimmermann [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219], but are written in a manifestly covariant form. The unitarity of the S matrix can be readily seen from the structure of these integral equations, and the explicit form of the S matrix can also be written down. The method is also extended to composite particles, but the case of unstable particles is not discussed. The equations possess divergence-free solutions, which are exhibited in the perturbation approximation. In particular, the equations are applied to quantum electrodynamics where the photon self energy is found to vanish and the polarization operator is obtained in a gauge invariant form.

M. J. Moravcsik (Upton, N.Y.).

Medvedev, B. V. On the construction of the scattering matrix. I. Integral causality condition in Bogoliubov's method. Soviet Physics. JETP **4** (1957), 671–680.

Following Bogoljubov the author introduces what could perhaps be called a "time dependent coupling constant" $g(x)$ and writes the S-matrix as a formal series

$$S(g)=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int S_n(x_1, \cdots, x_n)g(x_1)\cdots g(x_n)dx_1\cdots dx_n.$$

The coefficients $S_n(x_1, \cdots, x_n)$ in this expansion are not determined from a Hamiltonian or from equations of motion but their structure is studied with the aid of general principles, viz. correspondence, relativistic invariance, unitarity and causality. The first three of these conditions are formulated in an essentially conventional way while the author uses a new form of the causality condition called "the integral causality condition". He considers two space time regions G_1 and G_2 such that G_1 is situated "later than or space like with respect to G_2". (The author gives very detailed explanations of these concepts but it would carry us too far if we try to give all the relevant definitions here.) Further, he introduces two functions $g_i(x)$ such that $g_i\neq 0$ only if x lies in the region G_i $(i=1, 2)$. His causality principle then requires that what happens in G_2 is not influenced by what happens in G_1. Mathematically, this means that the total S-matrix $S(g_1+g_2)$ must be a product of $S(g_2)$ and $S(g_1)$ with the factor involving g_2 to the right:

$$S(g_1+g_2)=S(g_1)S(g_2).$$

This is the author's version of causality. He then shows that this condition implies both a similar condition in differential form used by Bogoljubov [cf., e.g., N. N. Bogoljubov and D. W. Shirkov, Uspehi Fiz. Nauk **55** (1955), 149–214=Fortschritte der Physik **3** (1955), 439–495] and the conventional postulate that the field operators commute for space like separations. As is further shown by detailed calculations the author can very easily obtain a certain standard form of the coefficients S_n above that has previously also been derived on the basis of Bogoljubov's differential form of the causality condition. [Cf. the paper quoted above.] *G. Källén.*

Gor'kov, L. P.; and Khalatnikov, I. M. Electrodynamics of charged scalar particles. Soviet Physics. JETP **4** (1957), 777–789.

The method of Landau, Abrikosov and Khalatnikov

[Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 497–500; MR **16**, 315] is applied to the problem of the determination of the asymptotic behaviour of the Green's functions of scalar electrodynamics in the region of high momenta. The Kemmer matrix formalism is employed, and the results obtained are qualitatively similar to those reported for spin $-\frac{1}{2}$ electrodynamics. In particular, as the cut-off parameter $\Lambda \to \infty$ the physical charge vanishes. The equivalence of this method with that of Gell-Mann and Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315] is shown for a special choice of gauge.

C. A. Hurst (Adelaide).

★ **Séminaire Théo Kahan, 3e année: 1955/1956. Physique des ondes.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+161 pp. (polycopiées)

This work consists of a collection of lecture notes presented at the seminar of T. Kahan by E. Arnous, A. Chevalier, and G. Rideau on the quantum theory of fields. The first chapter deals with the quantization of meson fields based on the space of Fock. Following this there is a rather complete discussion of Hilbert space and self-adjoint operators. Then comes a chapter on the axiomatization of quantum mechanics as developed by von Neumann and G. Ludwig. The last third of the notes takes up the problem of quantizing an electromagnetic field.

R. S. Phillips (Los Angeles, Calif.).

Vescan, Théophile T.; et Ioniță-Bârsan, Georgette. Remarques sur certaines corrections spectrales relativistes concernant les noyaux spatialement étendus. C. R. Acad. Sci. Paris **244** (1957), 2358–2361.

The first-named author has [Com. Acad. P. P. Romîne **6** (1956), 259–261; MR **18**, 362] derived the relativistic metric corresponding to a rapidly rotating electrically charged sphere. This metric is used to determine the possible energy levels of an atom consisting of such a sphere as a nucleus, and an electron. The result so obtained is then corrected, the correction term being derived from Dirac's equation. *A. Erdélyi* (Pasadena, Calif.).

Sitenko, A. G. On the theory of the stripping reaction. Soviet Physics. JETP **4** (1957), 492–496.

A new derivation of the cross section for (d, p) stripping reaction is given. It uses for incoming deuteron and outgoing proton wave functions which are modified by the nuclear potential field acting on these particles. The wave function of the proton-neutron system is assumed to be the product of the deuteron ground state wave function and the center of mass wave function, except when the neutron is inside the target nucleus, where the proton-neutron interaction is neglected. The differential cross section obtained reduces to the result of Butler [Proc. Roy. Soc. London. Ser. A. **208** (1951), 559–579] when deuteron and proton scattering in the nuclear field are neglected. In the limit of high incident energy the differential cross section agrees with the predictions of Serber's optical model [Phys. Rev. (2) **72** (1947), 1008–1016].

L. Van Hove (Utrecht).

Tassie, L. J. Inelastic scattering of electrons by nuclei. Nuovo Cimento (10) **5** (1957), 1497–1509.

Inelastic electron scattering is a way to obtain information about the excited levels of nuclei. The theoretical treatment of such scattering, however, is still in an elementary stage. The contribution of this paper is to point out that in general there is a different energy dependence of the differential cross section on energy for a $\Delta M=0$ and a $\Delta M=\pm 1$ transition, where M is the component of the nuclear spin in the direction of the vector change of momentum of the scattered electron. Then a specific theory (the independent particle model) is applied to $\Delta M=0$ transitions and the results are compared with experiments on ^{12}C. It is concluded that the independent particle model can explain the electron excitation of the 7.68 Mev level of ^{12}C, but cannot explain the excitation of the 4.43 Mev level. Apparently the full explanation of inelastic electron scattering must also include some collective effects of the nucleus. *M. J. Moravcsik.*

Grandmontagne, Raymond. Détermination approchée du "centre" de la fonction d'onde de Morse. C. R. Acad. Sci. Paris **244** (1957), 2586–2589.

It is shown that the central zero of the molecular wave function for the Morse potential is approximately at the value $r=r_e+(4/3)[\nu+(1/2)]$, for ν an odd integer, larger than 10. A number of other properties of the wave function are noted, which are valid for all ν. *P. M. Morse.*

Moffitt, W.; and Liehr, A. D. Configurational instability of degenerate electronic states. Phys. Rev. (2) **106** (1957), 1195–1200.

The implications of the Jahn-Teller theorem are discussed with special reference to the case where the forces tending to lower the symmetry of electronically degenerate molecular states are of the same order as the restoring forces encountered during typical vibrations. It is then impossible to separate the nuclear and electronic motions by means of the Born-Oppenheimer approximation. The resulting dynamical situation, which involves coupling between nuclear modes and low-frequency electronic motions, is analysed in detail for a specialized hexagonal model. The extension of this treatment to more general situations is discussed briefly.

A. C. Hurley (Melbourne).

Skorniakov, G. V.; and Ter-Martirosian, K. A. Three body problem for short range forces. I. Scattering of low energy neutrons by deuterons. Soviet Physics. JETP **4** (1957), 648–661.

The scattering of low energy neutrons by deuterons is considered by a generalization of the Bethe-Peierls treatment of the deuteron. In the limit of zero effective range, an integral equation is first derived for three identical particles, and elastic and inelastic scattering amplitudes discussed for the case in which two of the particles are bound. Spin and isotropic spin dependent central forces are then introduced, and the zero-energy neutron-deuteron scattering amplitudes determined to be $a_{3/2}=.51\times10^{-12}$ cm, $a_{1/2}=.3\times10^{-12}$ cm (both $\pm.17\times10^{-12}$). These depend weakly on the value of the singlet scattering length. The agreement with experiment of $a_{3/2}$ is good, but $a_{1/2}$ is so small as to make comparison inconclusive in that case, and requires the next term in the effective range expansion. *S. Deser* (Cambridge, Mass.).

Pais, A.; and Serber, R. Strong coupling. Phys. Rev. (2) **105** (1957), 1636–1652.

This long paper is methodological in character and deals in great detail with charged scalar meson theory with a static extended source. Some quite general transformations are given, valid in the fully relativistic case and for any coupling strength, which show up the isobaric nature of the nucleon spectrum. This transformation method is

then applied to the specific case stated above. The interaction is made proportional to τ_1 and the states corresponding to $\tau_1=-1$ and $+1$ are treated separately. For each of these the meson field is divided into a static and a nonstatic part, the former being chosen in such a way that it makes the major contribution to the energy. In the static approximation bound states exist for arbitrarily large values of the charge. A study of the non-static effects shows that the condition $g^2 \gg 1$ is not sufficient to ensure that the static approximation is a good one. Finally the question of scattering is discussed. The scattering in the limit $g^2=\infty$ (when the spectrum is completely degenerate) can be solved exactly, and the effect of finite isobar level separation is treated as a perturbation.

P. T. Matthews (Rochester, N.Y.).

Weinberg, Steven. Role of strong interactions in decay processes. Phys. Rev. (2) **106** (1957), 1301–1306.

The author considers the possibility of constructing a Lagrangian for β-decay, which is renormalizable to all orders in the pion-nucleon and electromagnetic coupling constants, provided the primary (weak) β interaction is only included to first order. In addition to the five primary β interactions (S, V, T, A, P) usually assumed, it is necessary to introduce nine other interactions. These are of the same nature as the $\lambda\phi^4$ term in the pion-nucleon interaction, and are required to supply counter terms to cancel the various types of primitive divergent which may occur. On this view, the rate of $\pi \to \mu+\nu$, for example, cannot be related to μ absorption by nuclei, but defines the renormalized coupling constant of one of the primary interactions. There is some comfort in the fact that the number of interactions is finite, but this still leaves more parameters in the theory than there are processes accessible to practical experiment. *P. T. Matthews.*

Watanabe, Shigueo. Polarization of particles with spin 1 in an elastic scattering at low energies for nuclei with spin zero. Rev. Mexicana Fis. **6** (1957), 59–71. (Spanish. English summary)

The polarization of a beam of spin one particles after an elastic scattering by spin zero nuclei at tow energy was studied. The transition matrix $M(\theta, E)$ is determined using Lepore's method, assuming the spin-orbit term in the Hamiltonian, but not the tensor force since the nuclei have spin zero. The result is that, if the incident beam is polarized the differential cross section of the elastic scattering is a sum of three terms. The first is independent of the incident and outgoing beams; the second term is a scalar product of the polarization vectors of the incident and outgoing beams, and the third one is the scalar product of the tensors of polarization of the two beams.

From the author's summary.

Stewart, A. L. Wave functions for helium and similar atomic systems. Proc. Phys. Soc. Sect. A. **70** (1957), 756–759.

Mayers, D. F. Relativistic self-consistent field calculation for mercury. Proc. Roy. Soc. London. Ser. A. **241** (1957), 93–109.

The method of self-consistent field approximation is used to calculate the normalized wave functions and other useful parameters for the mercury atom. It is found that the relativistic corrections are substantial, specially for the inner wave functions. The mathematical treatment, which involves extensive numerical integrations, is briefly discussed. *S. N. Gupta* (Detroit, Mich.).

Tauber, G. E.; and Wu, Ta-You. Self-consistent treatment of the independent-particle central-field nuclear model. Phys. Rev. (2) **105** (1957), 1772–1783.

This paper attempts to relate the average potential assumed in the shell model to a two body potential between nucleons. A two body interaction of Yukawa shape is assumed which may give reasonable agreement with the low energy two nucleon data, and still leaves the strength as an arbitrary parameter. This strength and the single particle wave functions are calculated to fit the total binding energy of O^{16} by the Hartree-Fock-Ritz method. The binding energies of O^{15} and O^{17} calculated with these parameters and wave functions, are much too small.

The binding energy of the "last" nucleon is also calculated and the central field in the Fock approximation. This has the wine bottle shape with a diffuse boundary. The potentials of the Hartree approximation, obtained by neglecting the exchange terms, are shown to be completely misleading in some cases. *P. T. Matthews.*

Huck, R. J. The effectiveness of variational methods for inelastic scattering problems. Proc. Phys. Soc. Sect. A. **70** (1957), 369–380.

The problem of 1_s-2_s excitation of atomic hydrogen by electron impact [Erskine and Massey, Proc. Roy. Soc. London Ser. A. **212** (1952), 521–530] leads to coupled integral-differential equations. Here artificial forms of kernel functions and potential energy functions give model equations which can be solved exactly. A detailed comparison is given with the results of variation methods by Hulthén [Kungl. Fysiogn. Sällsk. i Lund Förh. **14** (1944), no. 21; MR **6**, 111], Kohn [Phys. Rev. (2) **74** (1948), 1763–1772], Moiseiwitsch [ibid. **82** (1951), 753], Rubinow [ibid. **98** (1955), 183–187; MR **16**, 1187].

C. Strachan (Aberdeen).

Hope, J. Nuclear spin-orbit energy for oscillator wave functions. Phys. Rev. (2) **106** (1957), 771–774.

Matrix elements of the two-body spin-orbit interaction between two-nucleon states in LS coupling are expressed in a convenient form which is particularly suitable for oscillator wave functions. The interaction between an almost-closed shell and an external inequivalent nucleon is also considered. *Author's summary.*

Lane, A. M. Nuclear reactions. Rev. Mod. Phys. **29** (1957) 191–199.

This paper has been presented at the International Congress of Theoretical Physics, held in Seattle in September 56. It contains an accurate review of the situation of the theory of nuclear reactions. It is explained how many new experimental facts have led physicists to abandon a model based on a strong absorption of the nucleon in nuclear matter and to assume a free path comparable with nuclear dimensions.

This new picture has led to the introduction of the so-called "optical models" which have been successful in explaining the cross sections for nucleon-nucleon collisions in a very wide energy range. The very interesting application of the optical model to low energy reactions (cloudy crystal ball model) is discussed in detail.

Finally the situation of the statistical theory of reactions is analized in terms of the new "weak absorption" picture. *S. Fubini* (Genoa).

★**Murray, Raymond L. Nuclear reactor physics.** Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957. xi+317 pp. $10.00.

Written at first-year graduate student level. Chapter 1 surveys rapidly the physical assemblies, theoretical problems and mathematical approach. The remaining chapters cover the following topics: 2, neutron motion, cross-sections, flux, attenuation, etc.; 3, flux distributions and critical mass, using the diffusion equation and integral equations, and the Fermi age theory; 4, the heterogeneous reactor; 5, two-group theory and introductory matrix methods for multiple reflectors; 6, time-dependent conditions and instability; 7, temperature effects in multiplication; 8, reactor control; 9, the use of the Boltzmann equation; 10, neutron slowing and multigroup methods. References, problems and answers are given at the ends of chapters, appendices on mathematical topics and physical constants at the end. Perhaps anyone able to read comfortably the mathematical methods might benefit from a more careful discussion in places. The text is vivid, diagrams numerous and clear, worked examples throughout the text are helpful and give a sense of numerical magnitude. Paragraphs are brief and there is an adequate index. Parameters and symbols occur in great number and a summary of these with definitions would have been valuable. Two trivial misprints were noticed, one (p. 309) in the Laplace transform of df/dt and (p. 308) the surprising statement 0°C=−273.15°K.

C. Strachan (Aberdeen).

Weinberg, A. M. Nuclear reactor physics. Nuovo Cimento (10) **3** (1956), supplemento, 1278–1323.

This is a two chapter article dealing only with those aspects of nuclear reactor physics which are essential to the understanding of chain reactions. In the first chapter four nuclear processes important to the operation of a chain reactor are discussed. A summary of the formulation of the cross sections relevant to reactor calculations is presented and pertinent definitions such as that of the criticality constant are given. The second chapter treats the following topics. (i) The First and Second Fundamental Theorems which state respectively that "The stationary neutron distribution in a critical bare uniform reactor is separable in space and energy" and that "The fast neutron non-leakage probability is the Fourier Transform of the slowing-down kernel". (ii) The applications of the two above theorems to the derivation of the critical conditions for the all thermal reactor, very large reactor, and many-group reactor. (iii) The neutron behavior as criticality is approached in a subcritical reactor. (iv) The mechanics for calculating critical conditions such as the slowing down density and the neutron energy distribution in non-thermal reactors. (v) Non uniform reactor systems and the appropriate neutron distribution equations. The chapters conclude with general remarks concerning microscopic reactor theory and reactor kinetics including the subject of delayed neutrons.

R. M. Davis (Washington, D.C.).

Fröman, Per Olof. Alpha decay of deformed nuclei. Mat. Fys. Skr. Danske Vid. Selsk. **1** (1957), no. 3, 76 pp.

A detailed analysis is given of the theory of alpha-particle emission from deformed (non-spherical) nuclei. The nuclear wave functions are obtained using the collective model of A. Bohr and B. R. Mottelson [Danske Vid. Selsk. Mat.-Fys. Medd. **27** (1953), no. 16]. Intensities, lifetimes, and angular distributions of alpha particles are calculated. Appendix B deals with the penetration of an alpha particle through an anisotropic potential barrier making use of a three-dimensional extension of the WKB method.

D. Falkoff (Waltham, Mass.).

Kar, K. C.; and Banerjee, S. P. On the nuclear shell structure. Indian J. Theoret. Phys. **4** (1956), 21–32.

Drukarev, G. F. The theory of collisions of electrons with atoms. Soviet Physics. JETP **4** (1957), 309–320.

The collision between an electron and an arbitrary atom is considered. The combined wave function, atom + electron is described by the coordinate and spin functions of this system, being built up from the atomic and one-electron functions to possess the required anti-symmetry. A system of integro-differential equations, similar to the Fock self-consistent equations, have been obtained for the one-electron functions; these equations can be transformed into integral equations. The angular variables have been separated and integral equations are obtained for the radial functions. These equations can be simplified if approximate atomic functions are used. The problem then reduces to a system of Volterra equations, suitable for calculations. An analysis of the asymptotic expressions is carried out and formulas for the effective cross sections are derived.

P. M. Morse.

Gupta, K. K. Non-relativistic approximation of Bhabha's equations. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 336–340.

The author expresses a relativistic field equation, which corresponds to particles of spin $\frac{3}{2}$ as well as spin $\frac{1}{2}$, in the Rarita-Schwinger form. He then finds the non-relativistic approximation of the field equation in an external magnetic field, and thus obtains the magnetic moments of the particles associated with the field.

S. N. Gupta.

Levinger, J. S.; Rustgi, M. L.; and Okamoto, K. Relativistic corrections to the dipole sum rule. Phys. Rev. (2) **106** (1957), 1191–1194.

Urin, M. G.; and Mokhov, V. N. Polarization of relativistic protons in Coulomb scattering. Soviet Physics. JETP **4** (1957), 738–740.

Perturbation theory was used to determine the polarization of relativistic protons which experience Coulomb scattering.

Authors' summary.

Yang, C. N. Present knowledge about the new particles. Rev. Mod. Phys. **29** (1957), 231–235.

Two possible clues to understanding are (1) experiments consistently confirm classification of interactions into strong interactions, electromagnetic ones only slightly weaker and vastly weaker interactions: (2) symmetry concepts, the subject of this paper. The "heavy particle quantum number" is defined and, for strong and electromagnetic interactions, the "strangeness" and isotopic spin. Among problems of current theoretical interest are: the relation of the θ^0 particle to its antiparticle, their methods of decay, predicted from symmetry principles; methods of determination of spin of strange particles; hyperfragments, i.e. nuclei with Λ^0 bound to them; "freak" particles. Finally a very detailed discussion is given of the relation between θ^+ and τ^+; various solutions of the parity problem are discussed including the violation of conservation of parity and possibly of time-reversal- and charge-conjugation-invariance.

C. Strachan.

de Groot, S. R.; and Tolhoek, H. A. On K-mesons. Nederl. Tijdschr. Natuurk. 23 (1957), 121–130. (Dutch)

Survey of the properties and problems of K-mesons, discussing the following subjects. The phenomenological classification of elementary particles; the θ-τ-problem and the non-conservation of parity for weak interactions; the hypothesis of Gell-Mann and Pais concerning the θ^0-particle. *N. G. van Kampen* (New York, N.Y.).

Markov, M. On dynamically deformable form factors in the theory of elementary particles. Nuovo Cimento (10) 3 (1956), supplemento, 760–772.

The author discusses difficulties and inconsistencies which arise when one attempts to introduce form factors, i.e. extended sources rather than point particles, in the theory of elementary particles in a relativistically invariant way. *D. Falkoff* (Waltham, Mass.).

Terletskii, Ia. P. On a rational system of symbols for fundamental particles. Soviet Physics. JETP 4 (1957), 575–576.

The following symbol is suggested for an elementary particle ${}^{\lambda}_{n}Z^{E}_{s}$, here n is nucleon-number, E is electric charge, λ is the lepton-number, and s is connected with "strangeness number" by the relation $s=\varepsilon+n-E$. Thus the proton, for example, is written as ${}_1p^+$ and Σ^+ as ${}_1\Sigma^+_1$. *A. Salam* (London).

Rozental', I. L. A quasi-unidimensional interpretation of the hydrodynamic theory of multiple particle production. Soviet Physics. JETP 4 (1957), 217–224.

The hydrodynamic theory of multiple particle production developed by L. D. Landau is based on introducing two stages of liquid breakup — unidimensional motion and conical breakup. The validity range of the second stage is exceedingly difficult to estimate. This work investigates a variant of hydrodynamic theory, in which only the unidimensional stage is considered. This variant is shown to give very good approximation for final temperatures $T_k \sim 1.5\mu - 2\mu$. At $T_k \sim \mu$ the unidimensional approximation (particularly for slow secondary particles) gives a result that is merely of the right order of magnitude. The dependence of the fastest particle on T_k is also investigated. It turns out that the condition $T_k \lesssim \mu$ must be satisfied for the calculated velocity to agree with the experimentally-observed one. This leads to the preliminary deduction that when $T_k \sim \mu$ the interaction cross section of the secondary particles (apparently pi-mesons) is of the same order of magnitude as the geometric cross section. *Author's summary.*

Bellman, Richard; Kalaba, Robert; and Wing, G. Milton. On the principle of invariant imbedding and one-dimensional neutron multiplication. Proc. Nat. Acad. Sci. U. S. A. 43 (1957), 517–520.

Consider a finite one-dimensional rod of length x constituted of a material with the property that a neutron traversing an infinitesimal interval $(y+dy, y)$ has the probability ady of splitting into two neutrons one going to the left and one to the right. Let a neutron enter the rod from the left. What is the number of neutrons emerging from the left (reflected neutrons) and the number of neutrons emerging from the right (transmitted neutrons)? Introducing the functions $p_n(x)$ (and $q_n(x)$) representing the probability that n particles will be reflected (transmitted) from an interval of length x over all time as a result of one neutron entering at x at time $t=0$ and considering the effect of an additional material of length Δx added at x, the authors show that

$$p_n(x+\Delta)=(1-a\Delta)\{p_n(x)(1-a\Delta)^n+ a\Delta\sum_{k=1}^{n} kp_k(x)p_{n-k}(x)\}+a\Delta p_{n-1}(x).$$

By letting $\Delta\to 0$ the limiting equations

$$p_n'(x)=-(n+1)ap_n(x)+ap_{n-1}(x)+a\sum_{k=1}^{n} kp_k(x)p_{n-k}(x), \quad (n\geqq 1)$$

$$p_0'(x)=-ap_0(x),$$

are derived. Similarly, it is found that

$$q_n'(x)=-anq_n(x)+\sum_{k=1}^{n}{}'kq_k(x)p_{n-k}(x) \quad (n=1, 2, \cdots).$$

If

$$u(x, n)=\sum_{n=0}^{\infty} p_n(x)r^n, \quad v(x, r)=\sum_{n=1}^{\infty} q_n(x)r^n,$$

the foregoing equations lead to

$$u_x=au(r-1)+aru_r(u-1),$$

$$v_x=ar(u-1)v_r.$$

The authors show how the foregoing could be extended to more general situations and problems. *S. Chandrasekhar* (Williams Bay, Wis.).

Behrends, R. E.; and Fronsdal, C. Fermi decay of higher spin particles. Phys. Rev. (2) 106 (1957), 345–353.

In this paper projection operators for arbitrary spin particles are constructed explicitly using a method due to Fronsdal. The theory is then applied to investigate the Fermi decay lifetimes of the muon and hyperons under different assumptions concerning their spin. The following results are obtained: a) The energy spectrum in muon decay can be reproduced with appropriate choices of the coupling constants both for spin $\frac{1}{2}$ and $\frac{3}{2}$ of the muon. b) Under the hypothesis of universel Fermi interaction the hyperons will have alternate modes of decay involving leptons with lifetimes rather longer than the one of the normal modes of decay. *S. Fubini* (Genève).

See also: Lie Groups and Algebras: Trifonov. **Statistical Thermodynamics and Mechanics:** Stratonovich; Brueckner and Sawada; Lee, Huang and Yang; Oppenheim and Ross; Ezawa, Tomozawa and Umezawa. **Structure of Matter:** Mattis.

Relativity

★ **Jubilee of relativity theory, Bern, July 11-16, 1955. Proceedings: edited by André Mercier and Michel Kervaire. Helvetica Physica Acta, Supplementum IV.** Birkhäuser Verlag, Basel, 1956. 286 pp. S. Fr. 36.

This conference had 89 participants and 38 papers were read of which 25 were short communications, 10 were main lectures, one was a summary by W. Pauli and two were jubilee orations. Of the first group, 9 papers run to 1.5 pages or less, 10 to between 1.5 and 3 pages and 6 to between 3 and 5. Of the main lectures, one, by W. Baade is not printed and the remainder run to an average length of 13 pages. The summary and the two orations have an average length of 7 pages. Brief discussions are given after most of the papers.

The purpose of the meeting is first revealed on p. 244

in the lecture by M. Born on Physics and Relativity. The intention was to "survey the consequences of relativity for the whole of physics". This lecture is probably the most valuable item in the book, for it gives an account by a friend and contemporary of Einstein of the early background of his work, of the contributions by other workers and of Einstein's views on relativity and the quantum theory. Of nearly equal interest is Kollros' oration entitled "Albert Einstein en Suisse: Souvenirs".

The brevity of most of the papers makes them little more than lengthy abstracts, followed by references to the original papers. Thus the volume would have been a valuable guide to the literature of relativity had a detailed subject-matter index, and an author index, been supplied.

One group of speakers sees relativity as a branch of pure mathematics, the lecture by Lichnerowicz on the general problems of integration of the equations of relativity being the chief contribution. The problem of Cauchy is discussed in general terms but no specific solutions are given, nor is it indicated that any such could be obtained by the methods used. Alexandrov on an axiomatic approach to relativity, Tits on homogeneous and isotropic spaces, de Beauregard (p. 72) on Lorentz transformations and quantum mechanics, Mme. Fourès-Bruhat on Cauchy's problem in relativistic electrodynamics, Geheniau and Debever on the fourteen curvature invariants of a 4-dimensional Riemannian space, Scherrer on vector fields in linear field theory and Mme. Tonnelat on the solutions of the equations of Einstein's last field theory, all deal with problems in pure mathematics. No clue is given as to what, if any, bearing these investigation have on physical questions. The lecture by Jordan on his theory of a variable gravitational constant and the paper on the same topic by Ludwig and Just, may also be included in this group. They provide interesting exercises in tensor calculus, differential geometry and partial differential equations but at no point is any experimental or observational evidence forthcoming to support the notion that the constant of gravitation is indeed a variable.

Another group of papers deals with specific matters that have, or could have, physical or astronomical implications. The lecture by Trumpler on the observational determination of the bending of light-rays and on the gravitational red-shift and that by H. P. Robertson on theoretical and observational cosmology are important contributions. The short papers in this category are v. Dantzig's analysis of the notion of metric in mathematical physics, Møller on time in general relativity, Heckmann and Schucking's new model universe in Newtonian cosmology, McCrea on circular orbits in the Schwarzschild field, Rosen on cylindrical gravitational waves and Fock (p. 204) on rotation in general relativity.

A third class of papers — all of them brief — contain qualitative or speculative opinions. Their authors are: de Beauregard (p. 99), Papapetrou, O. Klein (p. 147), Hoyle, Bondi, Pirani and Fock (p. 239).

But the main topic that engaged the attention of the conference was the perennial question that occupied Einstein himself for so many years: How can a unified field theory of gravitation and electromagnetism, or, alternatively, a unification of general relativity and quantum mechanics be set up? Four main lectures were devoted to these matters. O. Klein (p. 56) spoke on generalisations of Einstein's theory of gravitation and quantum field theory, P. G. Bergmann on the quantising of general covariant field theories, Wigner on the relativistic invariance of quantum-mechanical equations and B. Kaufmann on the mathematical structure of the non-symmetric field theory. Shorter papers were provided by Corinaldesi and by Infeld. This reviewer cannot but admire the perseverance of those who continue to believe that such enquiries can be successful, in spite of the repeated failures since the year 1919. Wigner alone ventures to approach the heart of the matter and to hint that, whereas special relativity undoubtedly forms a part of micro-physics, gravitation as interpreted by general relativity belongs to macro-physics and that there may be no connecting link between the two. In this he appears to be supported by Born (p. 225). Incidentally, it is interesting to note the reaction of Born, a physicist, to general relativity. On p. 253 he implies that he was so repelled by this theory that he decided never to work in it. Yet, on p. 225 he confidently rejects as unproven observationally the bending of light-rays and the gravitational red-shift. But the astronomer Trumpler, accustomed to working in macro-physics and knowing to what extent a theory can be checked by astronomical observation, accepts both effects as observationally verified. The fact that general relativity is the only field theory that allows the equations of motion of a particle to be deduced from the field equations, is adduced by Bergmann as a reason for continuing the struggle to harmonize quantum theory with general relativity. It may equally well be argued that this property of general relativity exists precisely because the theory is concerned with the macroscopic phenomenon of gravitation; and that it does not hold for the other field theories just because they deal with non-gravitational forces. At any rate, all four lectures register the continuing lack of success in setting up the mathematical framework which will provide the unification and they also fail to suggest what empirical data, now inexplicable, would be elucidated if success could be achieved.

G. C. McVittie (Urbana, Ill.).

Bargmann, V. Relativity. Rev. Mod. Phys. **29** (1957), 161–174.

This is a valuable concise review of the present position of the subject. It is divided into 4 parts: (i) Special Relativity; (ii) General Relativity; (iii) Unified Field Theories; (iv) Quantization of Covariant Theories.

(i) Bargmann makes the point [which is of great historical signifiannce in view of the peculiar claim on their behalf made by E. J. Whittaker, A history of the theories of aether and electricity, vol. II, Philos. Library, New York, 1954; MR 15, 769] that, whereas Lorentz and Poincaré put electrodynamics in the foreground, Einstein used only on the basic laws of light propagation in developing special relativity. Moreover, recent work has shown that this theory is not restricted to electrodynamics. Bargmann discusses the experimental verification of this theory and its significance for quantum theory.

(ii) The basic principles of general relativity and the field equations of gravitation are clearly and carefully discussed. Particular attention is paid to the test-particle and n-body problems, mathematical investigations by Lichnerowicz and Wheeler of the free field equations, and the astronomical tests.

(iii) Discussion of unified field theories is confined to the five-dimensional theory of Kaluza and Klein, including its quantum generalization by Klein, the Jordan and Dirac theories involving a variable gravitational "constant", and the non-symmetric theories of Einstein and Schrödinger.

(iv) A few general remarks are made on the work of P. G. Bergmann and his students.

In discussion arising out of the paper, H. P. Robertson suggested that the Kennedy-Thorndike experiment (similar to Michelson-Morley but with the two arms along which light is propagated as unequal as feasible) should be repeated. *G. J. Whitrow* (London).

Arzeliès, Henri. Emploi de la transformation de Lorentz pour des vitesses relatives de référentiels supérieures à c. C. R. Acad. Sci. Paris **244** (1957), 2698–2700.

The author suggests the possibility of applying Lorentz transformations for relative velocities greater than that of light to solve some simple problems. However, no physical interpretation is given to the imaginary momentum vector which results. *A. Trautman* (Warsaw).

★**Synge, J. L. The relativistic gas.** North-Holland Publishing Company, Amsterdam; Interscience Publishers Inc., New York, 1957. xi+108 pp. $4.50.

The author's stated purpose in writing this book is "to develop in a simple way some formulae for a relativistic gas, that is, a system of material particles (and perhaps photons) which by elastic collisions between them take up certain equilibrium distributions." Although the main results are not new the exposition is clear, precise, and well written. The book is a supplement to a recent book on the special theory of relativity by the same author [Relativity: the special theory, North-Holland Publ. Co., Amsterdam, 1956; MR **17**, 1013] with the same notation and the same emphasis on Minkowskian geometry.

After three introductory chapters, the author determines in Chapter IV the most probable distribution function. He shows that for a simple gas consisting of material particles of proper mass m the distribution function is

$$N=C\exp(\xi_\tau M_\tau),$$

where C and ξ_τ are functions of events in space time such that if

$$N_\tau=C\int M_\tau\exp(\xi_p M_p)dw,$$

$$T_{\tau s}=C\int M_\tau M_s\exp(\xi_p M_p)dw,$$

then the five conservation equations,

$$\frac{\partial N_\tau}{\partial x_\tau}=0,\qquad \frac{\partial T_{\tau s}}{\partial x_s}=0,$$

are satisfied. The integrations in the above equations are carried out in an invariant manner over the hyperboloid

$$M_\tau M_\tau=-m^2$$

in four-momentum space. The corresponding results for a simple gas composed of photons (particles with $m=0$) and for a mixture of three simple gases, $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ are obtained where $G^{(1)}$ is composed of material particles of proper mass $m^{(1)}$, $G^{(2)}$ is composed of material particles of proper mass $m^{(2)}$, and $G^{(3)}$ is composed of photons.

Chapter V is devoted to a discussion of temperature, most probable speeds and distribution of energy; Chapter VI discusses first integrals of the conservation equations and Chapter VII discusses shock waves. Included in this last chapter is a discussion of the proper entropy per unit mass and its relation to the entropy integral defined in terms of the distribution function.

The book also contains in an appendix a derivation of L. H. Thomas' equations for the radiation field in a moving gas [Quart. J. Math. Oxford Ser. **1** (1930), 239–251]. *A. H. Taub* (Urbana, Ill.).

Clemmow, P. C.; and Willson, A. J. A relativistic form of Boltzmann's transport equation in the absence of collisions. Proc. Cambridge Philos. Soc. **53** (1957), 222–225.

The relativistic corrections to Boltzmann's equation are obtained in the form of appropriately placed β-factors ($\beta=[1-v^2/c^2]^{\frac{1}{2}}$) and by referring the density f to the configuration space (x_1, x_2, x_3) of a particular observer, as well as to the "reduced velocity" space (u_1, u_2, u_3) of the same observer. "Reduced velocity" is displacement per unit proper time (but for a factor c incorporated into the definition by the authors).

The relativistic Boltzmann equation given in the paper reduces to

$$\beta^{-1}\partial f/\partial t+u_\lambda\partial f/\partial x_\lambda+m^{-1}\beta^{-1}\partial(fF_\lambda)/\partial u_\lambda=0,$$

omitting factors c. Here m is the particle rest-mass and F_λ are components of force (change of momentum per unit observer's time). It is shown that f is Lorentz-invariant: there is a cancellation of β-factors in the 6-fold volume element $d^3x d^3u$ since in configuration space $\beta^{-1}d^3x$ is the invariant element while in reduced velocity space it is βd^3u owing to the condition that u_1, u_2, u_3 are components of a vector of fixed interval. [See also Synge, Relativity, Interscience, New York, 1956; MR **17**, 1013; and the book reviewed above.] The authors show, further, that $\beta^{-1}\partial(fF_\lambda)/\partial u_\lambda$ is invariant so that the stated equation is invariant. *O. Buneman* (Stanford, Calif.).

Tulczyjew, W. On the energy-momentum tensor density for simple pole particles. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 279–282, XXIII. (Russian summary)

The author assumes the energy-momentum tensor for a system of simple pole particles. Such a system is proportional to δ functions: $T^{\alpha\beta}=\sum_a{}^a t^{\alpha\beta}\,{}^a\delta$. It is shown in this paper that the field equations of general relativity theory determine completely, without further assumptions, the form of $t^{\alpha\beta}$, giving for it the usual product of velocity vectors. *L. Infeld* (Warsaw).

Bażański, Stanisław. The equations of motion of charged particles in general relativity. Acta Phys. Polon. **15** (1956), 363–379 (1957). (Russian summary)

Equations of motion for charged particles in general relativity theory were obtained for the first time by P. R. Wallace [Amer. J. Math. **63** (1941), 729–740; MR **3**, 212]. The author investigates the same problem using the simplified method developed by Infeld [Acta Phys. Polon. **13** (1954), 187–204; MR **16**, 531]. In all essential terms the result is the same as obtained by Wallace and by Bertotti [Nuovo Cimento (10) **2** (1955), 231–240; MR **17**, 545]. Yet there are some differences in the less essential expressions. *L. Infeld* (Warsaw).

Hessaby, M. Modèle de particule infinie. J. Phys. Radium (8) **18** (1957), 323–326.

This paper appears to be an attempt to find the gravitational field due to a particle which has an electric charge and magnetic moment, the electromagnetic field being represented by a 4-potential with only two components, Φ^3 and Φ^4. But some of the basic formulae are unsatisfactory; the author seems to have gone astray in his use of contravariant components. Putting $\Phi^3=0$, he has a problem with spherical symmetry (field of a charged particle); he reconstructs the well-known solution [cf. A. S. Eddington, The mathematical theory of relativity, 2nd

ed., Cambridge, 1924, p. 185] and makes some calculations of energy. Then he reinstates Φ^3, and finds it by a curious dimensional argument, apparently forgetting the presence of Φ^3 destroys the spherical symmetry, so that the line-element is no longer applicable. However, the argument is obscure throughout, and this interpretation of it may not be correct. *J. L. Synge* (Dublin).

Frankl', F. I. On the correct setting of Cauchy's problem and properties of harmonic coordinates in the general theory of relativity. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 189–196. (Russian)

Fok, V. A. Remarks on the article by F. I. Frankl' "On the correct setting of Cauchy's problem and properties of harmonic coordinates in the general theory of relativity". Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 197–198. (Russian)

These two papers continue a polemical discussion between the two authors that has been going on for years. The bone of contention is a property of "harmonic coordinates" which seem to have been introduced by Fok many years ago; these are coordinates for which the relation $g^{\alpha\beta}\Gamma_{\iota,\alpha\beta}=0$ holds. Such coordinates seem to possess definite advantages, a statement that Frankl' agrees with; the question is whether when one uses them the gravitational equations uniquely determine the field (up to Lorentz transformations with constant coefficients); or, rather the question is whether it is possible to introduce some additional conditions under which the uniqueness will be assured. Frankl' shows that the various conditions indicated by Fok at different times are not sufficient; Fok seems to feel that such conditions can be given but he indicates that the problem has no physical significance as long as electromagnetic effects, which are numerically overwhelmingly more important, are neglected.

G. Y. Rainich (Ann Arbor, Mich.).

Shirokov, M. F.; and Brodovskii, V. B. On the equations of motion of finite masses in the general theory of relativity. Soviet Physics. JETP **4** (1957), 904–909.

The problem of n bodies in the gravitational theory of general relativity is considered. The object is to deduce the equations of motion of any one of the bodies from the field equations. In the course of the proof it is stated that the pseudo-energy-tensor of the gravitational field is a tensor (Eqn. (2.3)) and an "angular momentum tensor" is defined (Eqn. (2.4)) which is also in fact not a tensor for an arbitrary coordinate system. The derivation of the equations of motion is based on an unstated theorem on the law of motion of the center of mass of the system of bodies. The equations of motion obtained are the same as those derived by Fock, Einstein and others but the method is said to be simpler. *G. C. McVittie.*

Pirani, F. A. E. On the physical significance of the Riemann tensor. Acta Phys. Polon. **15** (1956), 389–405 (1957). (Russian summary)

The author seeks to avoid familiar confusions in the physical interpretation of general relativity by working with the invariant (physical) components of vectors and tensors relative to an assigned orthonormal tetrad. An appropriate notation is carefully set up. {Criticism: the suffixes a and α have different meanings, but these two letters are almost indistinguishable in the small type used for suffixes.} The choice of tetrad is arbitrary, but reasons are given for choosing, along a given world line, a tetrad which undergoes Fermi propagation. The equation of geodesic deviation, in invariant form, is used to discuss relative motion in a cloud of particles, with a Newtonian comparison. Fermi propagation of a tetrad is studied for a circular orbit C in the Schwarzschild field, in three cases: (i) C is a circular orbit in flat space time (this gives essentially the Thomas precession; (ii) C is described arbitrarily slowly (the change in direction results from the inertial drag of the central body, a purely general relativity effect); (iii) C is a geodesic (the preceding effects are combined). A simple model gyroscope is treated, following, with modification, A. Papapetrou and E. Corinaldesi [Proc. Roy. Soc. London. Ser. A. **209** (1951), 248–258, 259–268; MR **13**, 695], and the paper ends with a discussion of discontinuities in the Riemann tensor, corresponding to discontinuities in density and stress.

J. L. Synge (Dublin).

Trautman, A. Discontinuities of field derivatives and radiation in covariant theories. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 273–277, XXIII. (Russian summary)

The author develops a simple general discussion of discontinuities for a generally covariant field theory derived from a first order Lagrangian. Let ψ_A be N field variables, $L^{AB\rho\sigma}\psi_{B,\rho\sigma}$ the second derivative terms in the field equations, and $\varphi(x^u)=0$ a hypersurface S in space-time. Let $\Lambda^{AB}=L^{AB\rho\sigma}\varphi_{,\rho}\varphi_{,\sigma}$ and let M be the maximum rank of the matrix $\{\Lambda^{AB}\}$, at a given space-time event, for all possible orientations of S. Then if $M=N$ and $\{\Lambda^{AB}\}$ has rank M on S, the Cauchy problem formulated on S has a unique solution in the neighbourhood of S. Example: Klein-Gordon equation with non-null S. But if $M<N$, the field equations have not a unique solution for any S. Examples: Maxwell's and Einstein's equations.

If the ψ_A and their first derivatives are assumed everywhere continuous, then discontinuities in $\psi_{A,\rho\sigma}$ can appear only across surfaces S for which $\{\Lambda^{AB}\}$ has rank $<N$, that is, if the field equations are underdetermined, or if the rank of $\{\Lambda^{AB}\}$ is less than M. In the latter case S is called a characteristic surface, and in this case only do the discontinuities have physical meaning.

The author determines the discontinuities across a characteristic surface, which he associates with radiation, for the Maxwell field and for the pure gravitational field. In the latter example he recovers the result of the reviewer [Phys. Rev. (2) **105** (1957), 1089–1099] for the discontinuity in the Riemann tensor across a null surface in empty space-time. *F. A. E. Pirani* (London).

Mavrides, Stamatia. Identités de Bianchi et identités de conservation en théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **244** (1957), 2482–2484.

The author derives conservation relations for the Einstein-Schrödinger theory directly from the generalized Bianchi identities (for an arbitrary affine connection) written down by Cartan [Ann. Sci. Ecole Norm. Sup. (3) **40** (1923), 325–412], instead of deducing them from a variational principle [cf., e.g., Mautner and Schrödinger, Proc. Roy. Irish. Acad. Sect. A. **50** (1945), 223–231; MR **7**, 397]. *F. A. E. Pirani* (London).

Kustaanheimo, Paul. On the use of a gravitational vector potential in the relativity theory of Birkhoff. Ann. Acad. Sci. Fenn. Ser. A. I. no. **228** (1957), 23 pp. (1956).

Birkhoff's theory leads to certain gravitational singu-

larities for photons and also when continuous matter is replaced by discrete particles in the manner suggested by Barajas [Comisión Impulsora y Coordinadora de la Investigación Cientifica, Anuario 1945, pp. 25–30] and Graef Fernandez [Cosmos y Cience **1** (1954), 44–65]. The author shows that a model can be constructed for the gravitational interaction of discrete particles which avoids these difficulties if Birkhoff's tensor potential is replaced by a certain vector potential field p^ν. A momentum-time τ is defined by $m^\nu d\tau = dx^\nu$, where m^ν denotes 4-momentum. The gravitational force-vector $dm^\nu/d\tau$ is stipulated to be a polynomial in p^ν, a homogeneous polynomial of the first degree in $\nabla_\mu p^\nu$ and a homogeneous polynomial of the second degree in m^ν. To restrict the coefficients, it is assumed that for small relative velocities Newtonian gravitational theory holds as a first approximation. The gravitational red-shift formula of Einstein then follows, but arbitrary values are possible for the perihelion advance in the one-body problem and for the gravitational deflection of light. If the axiom that proper mass is constant along the world-line of any particle of the model is replaced by the weaker axiom that the velocity of light is a limiting velocity which is never attained by any material particle, no consequences can be drawn concerning the absolute magnitudes in the three Einstein tests (any values can arise depending on suitable choice of the parameters of the theory), but the ratio of the perihelion advancements of two planets, the ratio of be deviations of two light rays passing at different distances from different stars and the corresponding ratio of gravitational red-shifts are the same as in Einstein's theory and in Birkhoff's. *G. J. Whitrow* (London).

Schücking, E. Nichtstatische kugelsymmetrische Lösungen der Vakuumfeldgleichungen in der erweiterten Gravitationstheorie. Z. Physik **148** (1957), 72–92.

The author establishes an analogue to Birkhoff's proposition of general relativity in the case of Jordan's extended theory of relativity [P. Jordan, Schwerkraft und Weltall, 2nd ed., Vieweg, Braunschweig, 1955; MR **17**, 1014]. He proves that the Jordan's field equations for the four-dimensional line-element

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

with the gravitational invariant $\varkappa = \varkappa(r, t)$ on the light cone can be solved in the case of spherical symmetry only through Schwarzschild's metric. He gives also new solutions of the Jordan's equations for different special statements of the line-element. *T. P. Anđelić.*

See also: Physical Applications: Finzi. **Statistical Thermodynamics and Mechanics:** Ezawa, Tomozawa and Umezawa. **Optics, Electromagnetic Theory, Circuits:** Kohler; McCrea; Carini; Crupi; Totaro. **Quantum Mechanics:** Skyrme.

Astronomy

Klemperer, W. B.; and Baker, R. M., Jr. Satellite librations. Astronaut. Acta **3** (1957), 16–27.

A satellite, natural or artificial, moving in a closed orbit around the Earth, is in a state of equilibrium under gravitational attraction and centrifugal force. If the shape of the satellite is oblong, it experiences to a certain degree stability of orientation; namely, it tends to direct its major axis along the vertical. Any deflection from this equilibrium position causes pendular oscillations of the satellite. The frequency of such oscillations, called "librations", can be computed, given the shape and the orbital period of the satellite.

The paper under review is concerned with such computations. The interest for this arose in connection with the question as to whether it is possible on a satellite vehicle to utilize, for orientation purposes, instruments responsive to the above mentioned librations.

A satellite in the shape of an idealized dumbbell (which consists of two massive bodies connected by a rigid rod) is most susceptible to the stabilizing effect. Such a satellite performs $\sqrt{3}=1.732$ complete cycle swings during one tour along the orbit (for instance, for an orbital period of 90 minutes the period of librations is 52 minutes). This movement is very slow and extremely difficult to detect.

The Moon has the shape of a tri-axial ellipsoid, pointing its major axis toward the Earth. As a consequence of this the Moon shows us always essentially the same side. It is plausible that the Moon has attained this position due to some pendular effect. For a nearly spherical ellipsoid ($e \ll 1$) the stabilizing libration frequency is $(3/2)^{\frac{1}{2}}e$ times the orbital frequency. As a consequence of this, the period of Moon's physical librations is of the order of 2 to 3 years. This movement is again too slow for easy detection. On the other hand, these oscillations are obscured by two other perturbations which cause librations of amplitude from 6° to 8°. The most important of these has a monthly period and is caused by the fact that the Moon describes an elliptic orbit rather than a circular one, and rotates about a slightly inclined axis. The other has a daily period and is caused by the rotation of the Earth.
E. Leimanis (Vancouver, B.C.).

Salpeter, E. E. Stellar energy sources. Rev. Mod. Phys. **29** (1957), 244–254.

The principal theme of this excellent review article concerns thermonuclear reactions and their influence on stellar evolution.

Following a concise account of the relevant astrophysical observations, particularly those embodied in the H-R (Hertzsprung-Russell) diagram, the method of calculating the internal structure of a star is summarized. Roughly speaking, the luminosity is mainly determined by the atomic physics and kinetic and gravitation theory of the equilibrium equations, whereas nuclear physics (which gives the rate of energy production) determines the central temperature at which energy production keeps pace with luminosity.

For a fairly young main sequence star the hypothesis of uniform chemical composition with energy-production concentrated towards the centre works well. The author discusses some of the main sources of inaccuracy in stellar interior calculations, e.g. present deficiencies in theories of turbulence and convection. In the case of main sequence stars of unknown age, helium accumulating in the deep interior yields a slightly non-uniform chemical composition. Calculations for the sun neglecting this effect led to some discrepancies with observation, but recent calculations taking due account of it indicate that the sun is about 30 per cent more luminous now than when it was young and homogeneous. The finite width of the main sequence band on a H-R diagram for nearly stars of all ages is thought to be mainly due to variations in age (and hence in chemical inhomogeneity).

Present ideas on stellar evolution are then briefly explained: Beginning as a cold diffuse gas, a star slowly contracts gravitationally until the central temperature is

just right for the conversion of hydrogen into helium. The star then remains, as a rule, in a steady-state on the main H-R sequence for a long time until an appreciable part of the hydrogen in its deep interior has been converted. Subsequent evolution depends on the star's mass and possibly on its angular momentum. First, it may move in the red giant direction (increase in radius and luminosity, decrease in surface temperature) while the inner core contracts gravitationally. Later, high enough central temperatures may be reached for further nuclear reactions to occur.

The author discusses these various evolutionary phases. He also considers in some detail the associated thermonuclear reactions. He then goes on to outline the possible evolution of Population II red giants (e.g., those in globular clusters). Semiquantitative agreement of theoretical calculations with observation has recently been obtained, but the point is made that even the qualitative features of the evolutionary track of an individual star after it leaves the main sequence are quite sensitive to the star's mass. The development of a star of more than twice the solar mass is probably rather different from that of globular cluster stars. Nevertheless, there is some indication that younger and more massive stars in Population I also evolve away from the main sequence in the red giant direction.

In conclusion, the author discusses two speculative ideas concerning ultra-rapid thermonuclear reactions and the generation of supernovae. *G. J. Whitrow.*

Zeuli, Tino. Influenza di una lenta rotazione sull' equilibrio relativo di una massa gassosa stellare. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 577–585.

A nearly spherical perfect gas is in slow rotation under the assumption of independence of all variables of the longitude coordinate. The rotational equilibrium equation, the adiabatic law equation, and Poisson's equation are solved simultaneously for the first order terms in the angular velocity, giving the pressure, density, temperature, and shape of the surface. *R. G. Langebartel.*

Holmberg, E. R. R. A new explanation of the recession of the nebulae. Monthly Not. Roy. Astr. Soc. **116** (1956), 691–698 (1957).

As an alternative to current theories of world-structure which agree in interpreting the red-shifts in the spectra of the extragalactic nebulae as indicators of recessional motion, the author suggests a geometrical explanation for the illusion of a recessional velocity proportional to distance. The coordinates x, y, z, ct of the light-cone are taken as homogeneous coordinates of a projective space with a uniform elliptic metric, but instead of the relation

$$x^2+y^2+z^2-c^2t^2=0$$

which would lead to infinite distance between two points, according to the Cayley theory, the author assumes that the time-coordinate must be replaced by w, where $c^2t^2=R^2-w^2$, and R is the "radius of the universe". The components of the velocity 4-vector are then interpreted as the plane coordinates of this space. By appealing to Clifford's work on parallels in elliptic space, the author claims (by arguments which are difficult to visualize physically) that the principle of relativity would lead to the illusion that distant objects would seem to be receding at a rate proportional to the distance.

G. J. Whitrow (London).

Stumpff, P. Über die Projektion von beobachteten Schweifpunkten auf die Ebene der Kometenbahn. Astr. Nachr. **283** (1957), 245–248.

The points in the tail of a comet are considered to lie in the comet's orbital plane. The determination of their space positions from a photograph therefore calls for a projective transformation from the plane of the photograph to that of the orbit, with the center of the lens as the center of projection. Practical formulas suited for mass computation by electronic calculator are derived based on two successive rotations of the plane of the photograph.

J. A. O'Keefe (Chevy Chase, Md.).

Ramakrishnan, Alladi. A stochastic model of a fluctuating density field. Astrophys. J. **119** (1954), 443–455.

Ramakrishnan, Alladi. A stochastic model of a fluctuating density field. II. Astrophys. J. **119** (1954), 682–685.

A generalization of the results of previous work of the author (see the paper listed above) in the theory of the fluctuation of the brightness of the Milky Way in the case of a finite system. The following formula is obtained, defining the moments of the magnitude of brightness

$$\begin{aligned}\varepsilon\{I^n(t)\}=\{1+\tfrac{1}{2}n(n+1)\alpha^2\tau_0\}[1-\exp(-t)]^n\\ -n(n-1+t)e^{-t}(1-e^{-t})^{n-1}\alpha^2\tau_0\\ +n(n-1)e^{-2t}(1-e^{-t})^{n-2}\alpha^2\tau_0,\end{aligned}$$

where t is the diameter of the system. For $t\to\infty$ this formula gives the result obtained by the author in the preceding article.

A comparison is made between the case here considered of a continuous distribution of interstellar matter and the case of discrete clouds of optical density τ_*. It is shown that for $\tau_*=2\alpha^2\tau_0$ all moments of the magnitude $I(t)$ coincide for the cases indicated.

A new derivation is given for the well-known formula of Chandrasekhar and Münch [Astrophys. J. **112** (1950), 380–392; MR **12**, 644] for $\varepsilon\{I^n(t)\}$ for a discrete distribution of absorbing matter.

I. N. Minin (RŽ Astr. Geod. **1955**, no. 1048).

See also: Ordinary Differential Equations: Poincaré. **Relativity:** Shirokov and Brodovskiĭ.

Geophysics

Frische, Richard H.; and von Buttlar, Haro. A theoretical study of induced electrical polarization. Geophysics **22** (1957), 688–706.

A mathematical solution is obtained and numerically evaluated for determining the depth to a saturated aquifer when prospecting for ground water by induced electrical polarization. In a horizontally stratified earth model consisting of a nonpolarizable overburden and an underlying, infinitely deep polarizable layer, the induced-polarization potential difference for a Wenner electrode configuration is nearly independent of the resistivity contrast. The computation agrees with results from a laboratory model tank and with a field curve. This justifies confidence in the validity of the results obtained from the model tank on earth models too complex for computation.

Authors' summary.

Monin, A. S. On macroturbulent exchange in the earth's atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1956, 452–463 (1 plate). (Russian)

Characteristics of macroturbulence, which are similar to fine-scale turbulence characteristics are introduced.

The characteristics are determined empirically from the data on the synoptic chart in order of magnitude. Macroturbulence is shown not to be always a dissipative factor for general circulations. The spectrum of the zonal circulation oscillations of the atmosphere is analyzed.

M. D. Friedman (Newtonville, Mass.).

Gates, W. Lawrence. Hemispheric numerical forecasting with the barotropic model, and some remarks on boundary-condition error. J. Meteorol. **14** (1957), 332–342.

Krastanov, L. New contents of the criterion for turbulence in the atmosphere. Izv. Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk. Ser. Fiz. **5** (1955), 149–159. (Bulgarian. Russian and English summaries)

The equation for the vertical transport of a "substance" in a turbulent atmosphere is obtained by using the continuity equation for an incompressible medium and by expressing instantaneous values by means of the mean values and pulsations, thus:

$$\rho\left(\frac{\partial\bar{s}}{\partial t}+\bar{u}\frac{\partial\bar{s}}{\partial x}+\bar{v}\frac{\partial\bar{s}}{\partial y}+\bar{w}\frac{\partial\bar{s}}{\partial z}\right)=\rho\frac{d\bar{s}}{dt}-\frac{\partial(\sigma'w')}{\partial z}.$$

The last term is the local change with height of the turbulent flow of the quantity of substance obtained from unit volume by just turbulent exchange. The barred letters in the equation are the mean values of density, velocity and substance concentration; σ' is pulsation of substance density and w' is pulsation of vertical velocity component. If transportable substance is assumed to be the quantity of heat, it can be expressed as enthalpy or potential enthalpy. It is established that the values of the last member obtained thus represent the local change of the turbulent flow with the height. It is established that the difference between the local change in these turbulent flows with the height is equal to the characteristic value in Richardson's criterion. It is established that turbulent motion can originate or be maintained in the surface layer of air for arbitrarily small values of the dissipation.

M. D. Friedman (Newtonville, Mass.).

Ullmann, W.; und Maaz, R. Zur Frage der "scheinbaren" und "wirklichen Neigung". Z. Geophys. **23** (1957), 75–82.

In discussing the motion of a seismograph pier, the authors concern themselves especially with the tilt, the apparent tilt due to translational acceleration, and the horizontal rotation. They find the tangent of the horizontal rotation more amenable to averaging than the angle itself. They compare their formulas with less accurate approximations of Wiechert, and give upper bounds for the angles in the case that the earth motion is representable by a Fourier polynomial. *A. Blake.*

Hardtwig, E. Über die Entstehung der Mikroseismik. Z. Geophys. **23** (1957), 83–112.

Microseisms are considered as characteristic vibrations of the earth's crust, channeled in such a way as to accentuate the periods associated with the minima of the group velocities of the several modes. This treatment alleviates the need to find external forcing functions of the requisite periods and persistence in order to account for the observations. The crust is visualized as a free plate, in analogy to a slab of concrete on a yielding substratum. The author suggests that this analogy is good for the weak motions of microseisms though not for stronger earthquake waves. He obtains good order of magnitude checks of his theory with observed periods of microseisms. It appears that considerable departure from the free plate behavior could be admitted without carrying the results of these calculations beyond the range of the observations. *A. Blake* (Ann Arbor, Mich.).

Lehn, C. J. The main problem of geodesy and the inverse problem solved by vector methods. Geodaet. Inst. Skr. (3) **26** (1957), 47 pp.

The determination of final position from distance and azimuth and the inverse problem are developed by vectorial methods of differential geometry for geodesic curves on the terrestrial spheroid. The direct problem makes use of the Frenet formulae together with high order derivatives to obtain a series expansion in the distance as parameter, both of the chord vector between the two positions and the normal vector at the final position. The inverse problem is handled essentially by inversion of the series approximation of the direct. Numerical examples are given. *N. A. Hall* (New Haven, Conn.).

Hirvonen, R. A. Computation of triangulations on the ellipsoid by the aid of closed formulas. Bull. Géodésique 1957, 3–15.

It is proposed to calculate geodetic triangles (sides of the order of 0.01 radian) by closed formulas of spherical trigonometry. The radius of the sphere adopted is the mean radius of curvature within the triangle. Corrections which reduce to the ellipsoid are tabulated; they are negligible for triangles with sides less than 30 km.

The method is contrasted with the current scheme, i.e. calculating the sides from a plane triangle by Legendre's theorem; then calculating the coordinates on the ellipsoid by series expansions in $s\cos\alpha$, $s\sin\alpha$.

The advantage lies with the closed formulas, even when these involve the loss of 1 or 2 significant figures, provided that the computer obtains his trigonometric functions from special Taylor tables [Hirvonen, same Bull. **1953**, 369–392; MR **15**, 650], which utilize series methods to generate trigonometric functions from a small table. *J. A. O'Keefe* (Chevy Chase, Md.).

Marussi, Antonio. La coordination des systèmes géodésiques. Calcul et compensation des grandes triangulations en prenant en considération la forme du géoide. Bull. Géodésique 1957, 16–19.

This paper considers the projection of points of the geoid on the ellipsoid using geographic coordinates (latitude and longitude on the earth considered a spheroid or oblate ellipsoid of revolution). The Christoffel symbols of two-dimensional (Riemannian) space are given relative to the linear element or metric of the spheroid; expressions are given without derivations for the linear element or metric of the geoid, the scale of the projection of the geoid on the spheroid, integrability condition for the geoid in terms of the derivatives of the deflections of the vertical, the difference of mean curvatures of the geoid and spheroid; also given are the Christoffel symbols relative to the metric of the geoid, the tensor representing the difference between the Christoffel symbols of the two surfaces and corresponding formulas for geodesic curvature of the transformed curves. A bibliography of seven papers is included, two by the author.

P. D. Thomas (Washington, D.C.).

See also: Integral and Integrodifferential Equations: Melzak. **Elasticity, Plasticity:** Stoneley. **Fluid Mechanics, Acoustics:** Phillips.

OTHER APPLICATIONS

Economics, Management Science

Beckmann, Martin J. A demand curve for luxuries. Trabajos Estadist. **8** (1957), 23–27. (Spanish summary)

Consider a commodity of which at most one unit can be bought by any individual during a unit time interval and which has n quality grades with prices $p_1<p_2<\cdots<p_n$. Assume the existence of a non-negative random vector $t=(t_1, \cdots, t_n)$ of "thresholds" such that an individual with income y buys grade k whenever $t_k\leqq y-p_k$, $t_{k+j}>y-p_{k+j}$ $(j=1, \cdots, n-k)$. If $f(t)$ is the probability density of t, and $g(y)$ the frequency density of incomes, the demand of all individuals for grade k depends on $p_k, \cdots, p_n$ only and is equal to

$$D_k(p_k, \cdots, p_n)= \int_{y-p_n}^{\infty}\cdots\int_{y-p_{k+1}}^{\infty}\int_{0}^{y-p_k}\cdots\int_{0}^{\infty}\int_{0}^{\infty}g(y)f(t)dt_1, \cdots, dt_n.$$

The total demand for the top grade can then be approximated thus: $D_n(p_n)\simeq c_1-c_2\exp\{ap_n\}$ $(a>0)$, provided two additional assumptions are made: 1) the top grade can be bought only when income exceeds a certain level $q\geqq p_n$; and for such high incomes, $g(y)$ is proportional to y^{-h}, $h>1$ ("Pareto's law"); 2) the distribution function for t_n is exponential: $\Pr(t_n\leqq x)=1-b\exp\{-a(y-p_n)\}$. The paper gives references to earlier suggestions, of which the most recent ones were made by J. Aitchison and J. A. C. Brown [Rev. Econ. Studies **22** (1954–55), 35–46], M. J. Farrell [J. Roy. Statist. Soc. Ser. A. **117** (1954), 171–201] and N. Georgescu-Roegen [Quart. J. Econ. **68** (1954), 503–534]. *J. Marschak.*

Mayer, Hansjörg. Die Beitragsfestsetzung nach Tarifwechsel in der privaten Krankenversicherung. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 221–239.

Zwinggi, Ernst. Zur Prämien- und Deckungskapitalberechnung bei erhöhter Sterblichkeit. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 141–145.

The author studies the effect on premiums and reserves in life insurance of an increase in the force of mortality from μ_x to $\mu_x^{(\gamma)}$, where for all ages x

$$\mu_x^{(\gamma)}=(1+\gamma)\mu_x \text{ or } \mu_x^{(\gamma)}=\gamma+\mu_x.$$

Simple approximate relations are given. Thus, e.g., the new premium is in both cases expressed as a linear function of γ. *K. Medin* (Stockholm).

Jecklin, H. Reserveberechnung nach i-Gruppen. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 21–35.

A summary is given of several approximative valuation methods suggested by the author and his followers. Contrary to Lidstone's valuation method, the endowment assurances are sorted according to time elapsed. In principle, the retrospective value is considered and hyperbolic interpolation used. Several different methods are applied to a numerical example. *P. Johansen.*

Hüsser, Rudolf. Orthogonale Polynome mehrerer Veränderlichen und ihre Anwendung in der ein- und zweidimensionalen Ausgleichsrechnung. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 55–123.

Ausführliche Darstellung der Ausgleichung mittelst orthogonaler Polynome durch Verwendung Stielt'jeser Integrale. Es werden im 1. Teil notwendige und hinreichende Bedingungen für die Bestimmung der Koeff. orthogonaler Polynome gegeben. Die Theorie wird angewendet auf eine und zwei unabhängige Variablen. Die entsprechenden Ausgleichungsformeln werden sowohl für äquidistante als auch nicht-äquidistante Intervalle angegeben. Die Ausgleichung mittelst orthogonaler Polynome wird am Beispiel einer flächenhaften Ausgleichung schweizischer Sterbetafeln demonstriert.

W. Saxer (Zürich).

Müller, Nikolaus. Ein praktisches Verfahren der Bestandsentwicklung einer Personengesamtheit unter genauer Berücksichtigung des Neuzugangs. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 197–220.

Gegeben sei ein Bestand von ungleichaltrigen Aktiven, die alle mit dem gleichen Eintrittsalter und gleichen Versicherungsbedingungen in eine Pensionskasse eintreten, welche Invaliden-, Alters- und Witwenrenten gewährt. Der Verf. berechnet mit Hilfe der diskontinuierlichen Methode und einer Summenformel von Schöbe die vermutliche Anzahl von Invaliden, Altersrentnern und Witwen nach einer beliebigen Anzahl von Jahren samt den dazugehörigen Belastungen. Diese Untersuchung dürfte vor allem für praktische Zwecke nützlich sein.

W. Saxer (Zürich).

Knoche, Hans-Georg. Versicherungsmathematische Funktionen als rationale Funktionen der Sterbenswahrscheinlichkeiten. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 183–195.

Systematische Untersuchung versicherungsmathematischer Grössen als Funktionen der Sterbenswahrscheinlichkeiten und Diskussion ihrer Ableitungen. Beispielsweise können aus solchen Betrachtungen die Zeichenwechselsätze von Moser [Mitt. Verein. Schweiz. Versich.-Math. **9** (1914), 1–8], Lidstone [J. Inst. Actuar. **39** (1905), 209–252] und Schärf [Mitt. Verein. Schweiz. Versich.-Math. **42** (1942), 147–149] beweisen werden.

W. Saxer (Zürich).

Page, E. S.; and Muris, Shirley. The effect of departures from assumption when manufacturing to a specification. Operations Res. **5** (1957), 68–74.

Dimensions or other measurable characteristics of manufactured items generally vary according to some distribution. Items whose characteristics fall outside required specifications must be either scrapped or reworked. There is thus a loss associated with any unit which fails to meet specifications, and this loss is reflected in the cost per unit that is satisfactory. The aim of the manufacturer is to minimize the cost per satisfactory unit. The calculation of the level at which one should attempt to control the manufacturing process so as to minimize the cost per satisfactory unit can be carried out if the underlying distribution of the characteristic being measured is known. Frequently the assumption is made that this distribution is normal. The question arises as to whether this assumption is justified and whether it leads to correct conclusions. Although no general answer is given to this question, the authors study a special example numerically and graphically. Their calculations indicate that, in this example, fairly considerable departures from normality cause rather small changes in the optimum obtained when the normality assumption holds strictly. *Benjamin Epstein.*

McMillan, B.; and Riordan, J. A moving single server problem. Ann. Math. Statist. **28** (1957), 471–478.

The paper deals with the following situation. An assembly line carries items to be served and is moving with uniform speed. The server moves with the line while working; as soon as service is completed he transfers momentarily to the next item. It is assumed that the service time is exponentially distributed and that servicing is stopped after a certain time. Let T be the time available for servicing when work is started on the first item and denote by $p(k, T)$ the probability that k items can be serviced during the available time T. Let $B(t)$ be the distribution of spacings between items on the assembly line and denote by $P(x, T)=\sum_{k=0}^{\infty} p(k, T)x^k$ the generating function of the $p(k, T)$. The authors state that $P(x, T)$ satisfies the relation

$$P(x, T)=\exp\{-\alpha T[1-x\int_0^{\infty} P(x, T)dB(T]\},$$

where α is the parameter of the service time distribution. They derive this relation under two alternative assumptions, namely, (a) $B(t)$ is a degenerate distribution, (b) $B(t)$ is an exponential distribution. *E. Lukacs.*

Vazsonyi, Andrew. Economic-lot-size formulas in manufacturing. Operations Res. **5** (1957), 28–44.

This paper generalizes the conventional economic lot size formula for a single part to a situation where a given set of requirements must be met simultaneously for many parts. Plans are to be made for future production periods. It is assumed that the available productive hours on each machine are known, that the time it takes to manufacture each part (including set-up time) is known, and that the cost of each set-up and cost of carrying inventory is known. The problem is to determine order quantities for each part so that production requirements are met and the combined cost of set-ups and inventory is minimized. It is shown that one is led to a very large non-linear programming problem which presents such difficulties that one cannot at present give a general solution to it. However relatively good feasible solutions (i.e., ones meeting requirements and staying within available machine capacities) that do not necessarily minimize combined costs can be obtained. Such feasible solutions can be programmed fairly simply on an electronic computer.

The author emphasizes in his concluding remarks that this is only a "modest step in extending the conventional economic lot-size formula by including some of the capacity limitations of production. However, we are still a long way from what one might call the ultimate economic lot-size formula". Such an ultimate formula needs to take into consideration not only the cost of carrying inventories and the cost of set-ups, but also such things as: (i) the sequence in which parts must be manufactured, since the realizability of a schedule may depend in an essential way on the particular sequence to be followed; (ii) the stochastic nature of demand; (iii) the stochastic nature of production due to such things as machine breakdowns, unavailability or lateness of tools and materials, absence of workers, etc. In other words, waiting line problems of a very complex form; (iv) changes in the cost function so as to include not only the cost of carrying inventories and cost of set-ups, but also to include costs reflecting the desirability of schedules calling for steady work loads, constant working force, etc.

Benjamin Epstein.

Supnick, Fred. Extreme Hamiltonian lines. Ann. of Math. (2) **66** (1957), 179–201.

The author considers the problem of determining the shortest (and longest) Hamiltonian line ("H-line") of a rectilinear graph, whose vertices are n distinct points of E_r and whose edges are rectilinear segments joining all distinct pairs of vertices, where an H-line is an "edge-chain" having each vertex of the graph as an end point of two and only two edges of the chain. This problem, and many others of its kind, are of considerable importance in management science and industrial engineering applications. They are as yet unsolved in the sense of finding either a reasonably short, or a most short, computational algorithm for determining the H-lines in specific numerical cases. He shows that if certain conditions on the vertices are assumed to hold then an explicitly stated ordering of the vertcies yields the H-lines. He also constructs various classes of concrete realizations of such conditions and thereby provides classes of graphs for which the H-lines are known explicitly; he is more successful in constructing classes explicitly stated for shortest H-lines than for longest H-lines. Note is made of several open problems and a few examples are included to show some of the limitations of his methods. *M. Flood.*

Vidale, M. L.; and Wolfe, H. B. An operations-research study of sales response to advertising. Operations Res. **5** (1957), 370–381.

In the analysis of advertising campaigns the authors describe the interaction of sales and advertising in terms of three parameters: 1) The sales decay constant λ defined as the exponential rate of decrease of sales of an unpromoted product; 2) the saturation level of sales M, defined as the sum of actual sales S and potential sales $M-S$; 3) the response constant r, defined as the sales generated per advertising dollar when sales are at $S=0$. In general, sales generated per advertising dollar when sales are at level S is given by $r(M-S)/M$.

The properties of each one of these three parameters are demonstrated with the help of diagrams for real data, brought from the wide practical experience of the authors.

The following mathematical model is formulated

$$dS/dt=rA(t)(M-S)M-\lambda S$$

where S is the rate of sales at time t and $A(t)$ is the rate of advertising expenditure. "This equation has the following interpretation: the increase in the rate of sales dS/dt, is proportional to the intensity of the advertising effort A, reaching the fraction of potential customers $(M-S)/M$, less the number of customers that are being lost λS."

This paper gives a convincing picture of the power of modern analytical market research. Combining operations research of this kind and motivations research, it would seem that the optimal design of an advertising campaign could nowadays be carefully determined. *S. Thore.*

Luchak, George. The distribution of the time required to reduce to some preassigned level a single-channel queue characterized by a time-dependent Poisson-distributed arrival rate and a general class of holding times. Operations Res. **5** (1957), 205–209.

Previously the author [Operations Res. **4** (1956), 711–732; MR **18**, 707] discussed the determination of the transition probabilities (as explicit functions of the time) for the type of queuing system described in the title of the present paper, i.e., for the system $M/G/1$ where the ser-

vice-time distribution is a "mixture" of scaled chi-square distributions (perhaps with infinitely many components). The system will be Markovian if its "present state" is assessed by enumerating the phases of services still outstanding. In the present work the author continues his previous investigation by finding explicit formulae for the distribution of the length of a "busy period" initiated by a given number of phases outstanding at $t=0$ in terms of the modified I-Bessel function employed in his previous paper. Many of the formulae permit the input density to vary arbitrarily with the epoch, and it is stated that temporal variations in the rate of service could be incorporated in the calculations if desired. Despite the generality of the model, some of the explicit formulae are relatively simple in form and are likely to be very useful in practice. The author remarks that any solution to the busy period problem can readily be translated to yield a result about first-passage time to a lower state.

D. G. Kendall (Oxford).

Weinwurm, Ernest H. Limitations of the scientific method in management science. Management Sci. **3** (1957), 225–233.

Discusses the usefulness and range of validity of "the scientific method" in analyzing problems of management. Points out the disagreements in opinion as to the universality of the quantitative approach and of the use of probability in reaching managerial decisions. Suggests many situations where these methods are useful but questions their exlcusive use as guides in "high level" decisions. *P. M. Morse* (Cambridge, Mass.).

See also: Mumerical Methods: Kimura. **Computing Machines:** Grabbe. **Probability:** Buch. **Statistics:** de Vries.

Programming, Resource Allocation, Games

Galler, Bernard A.; and Dwyer, Paul S. Translating the method of reduced matrices to machines. Naval Res. Logist. Quart. **4** (1957), 55–71.

This paper gives a new method for solving the general transportation problem in k dimensions, the method of reduced matrices which is based on the fact that a constant may be subtracted from each element of any row (column, etc.) of the cost matrix without changing the positions in the matrix where the final allocations should be made. The authors adapt this method which terminates with a positive, but not necessarily integral, solution to a form suitable for machine computation and carry an example through all its stages. The adjustment of fractional solutions (which do not occur in the usual case $k=2$) is discussed but not illustrated. *P. Rabinowitz.*

Pearson, Carl E. Note on linear programming. Quart. Appl. Math. **14** (1956), 205–206.

A simple direct proof is given for the theorem: if there exist numbers $x_j \geq 0$ minimizing the form $\sum_{j=1}^{n} c_j x_j$ under the conditions $\sum_{j=1}^{n} a_{\alpha j} x_j = b_\alpha$, $i=1, \cdots, m$, then it is possible to find these numbers x_j such that not more than m of them are different from 0 and the columns of the matrix $\|a_{\alpha j}\|$ corresponding to non-vanishing x_j are linearly independent.

O. V. Šalaevskiĭ (RŽ Mat 1957, no. 1658).

Eisemann, Kurt. The trim problem. Management Sci. **3** (1957), 279–284.

The problem of minimizing the trim losses in cutting rolls of any material for the execution of business orders, is formulated, solved and discussed by applying the standard linear programming techniques.

A. G. Azpeitia (Providence, R.I.).

See also: Difference Equations, Functional Equations: Bellman. **Topological Vector Spaces:** Fan. **Convex Domains, Integral Geometry:** Goldman and Tucker; Heller. **Economics, Management Science:** Vazsonyi.

Biology and Sociology

Smith, C. A. B. Counting methods in genetical statistics. Ann. Human Genetics **21** (1957), 254–276.

A general method has been given by R. Ceppellini, M. Siniscalco and the author [same Ann. **20** (1955), 97–115; MR **17**, 761] in the estimation of gene frequencies in a large population. In the present paper the author generalizes the method to cover other genetical problems and to include the testing of goodness of fit and homogeneity as well as estimation. Applications are made especially in detail to the problems of linkage and of gene frequencies of MNS blood groups. Finally it is proved that the methods of counting and of maximum likelihood are equivalent.

Y. Komatu (Tokyo).

See also: Integral and integrodifferential Equations: Melzak. **Probability:** Katz and Powell. **Statistics:** Guttman.

Information and Communication Theory

Mooers, Calvin N. Choice and coding in information retrieval systems. Trans. I.R.E. **PGIT-4** (1954), 112–118.

A library classification system is assumed in which each item in the library is referred to by a tally (such as a card) containing binary coded information characterizing a set of "descriptors" which represent properties of the subject matter of the item in question. Information retrieval is accomplished by a machine which is supplied by the user with a set of the descriptors. The machine scans all the tallies and extracts those whose set of descriptors contains, as a subset, the set supplied by the user. It is pointed out that workable systems are possible in which all descriptors do not have distinct coded forms.

The superimposed coding system for descriptors uses a coded form on the tally consisting of the bitwise logical sum of the descriptors in the corresponding set. This coding system admits more rapid information retrieval but also results in occasional selection of unwanted tallies and requires more bits per tally. An information theoretical analysis shows that the effective information per tally bit is reduced by the factor $\log_e 2$ as a result of superimposing descriptors. *D. E. Muller.*

Tanner, Wilson P., Jr.; and Swets, John A. The human use of information. I. Signal detection for the case of the signal known exactly. Trans. I.R.E. **PGIT-4** (1954), 213–221.

Tanner, Wilson P., Jr.; and Norman, Robert Z. The human use of information. II. Signal detection for the case of an unknown signal parameter. Trans. I.R.E. **PGIT-4** (1954), 222–227.

As an alternative to the conventional threshold theory of sensory detection, a model based on statistical decision

theory is briefly discussed. The model is based on the assumption that the observer contains a hypothesis-testing mechanism which maximizes an expected gain. The papers are not mathematical in nature, but describe experiments made to test the theory, the first using visual detection, the second auditory detection. *V. E. Beneš.*

Devooght, J. Sur la loi de Zipf-Mandelbrot. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 244–251.

Les constantes de la distribution canonique de Mandelbrot et le nombre moyen de mots différents T (types) d'un texte de N mots (signes) sont calculés explicitement. On en déduit que la température informationnelle θ d'un langage à lexique infini est égale à l'ordre de croissance du nombre de types en fonction du nombre de signes. Une nouvelle méthode de calcul de θ est proposée. L'étude du comportement asymptotique des „échelons" de la courbe de Zipf conduit à une dérivation plus satisfaisante de la forme approchée de celle-ci. *Résumé de l'auteur.*

Feinstein, Amiel. A new basic theorem of information theory. Trans. I.R.E. **PGIT-4** (1954), 2–22.

See Feinstein, Res. Lab. Electronics, Mass. Inst. Tech., Tech. Rep. No. **282** (1954); MR **17**, 1098. *S. Kullback.*

Medina, Alejandro. On synthesis of linear systems for the prediction of chaotic signals. Rev. Mexicana Fis. **6** (1957), 73–86. (Spanish. English summary)

Die Wienersche Integralgleichung für die Greensche Funktion $S(t)$ eines Filters, auf die das Problem der Bestimmung eines linearen Filters mit minimaler quadratischer Fehlererwartung bei gegebenen ein- und ausgehenden Signalen führt, wird formal durch Approximation von $S(t)$ durch Linearkombinationen von Funktionen $\delta(t-t_k)$ behandelt, einem in anderen Zusammenhang von Cerrillo [Rev. Mexicana Fis. **4** (1955), 61–92; **5** (1956), 71–132] benutzten Verfahren entsprechend. Physikalisch entspricht dem eine Anordnung von "Fenstern". Über die stochastischen Prozesse, die die Signale darstellen, wird unter anderem vorausgesetzt, daß gewisse Kovarianzfunktionen hinsichtlich St) integrierbar sind.

K. Krickeberg (Würzburg).

Medina, Alejandro. On the use of window distributions for transformation of stochastic signals. Rev. Mexicana Fis. **6** (1957), 87–104. (Spanish. English summary)

Es wird erörtert, wieweit die Voraussetzungen über die Signale, die dem im vorangegangenen Referat geschilderten Verfahren zugrunde ligen, abgeschwächt werden können. *K. Krickeberg* (Würzburg).

Wilkinson, Roger I. Theories for toll traffic engineering in the U. S. A. The Bell System Technical Journal Vol. 35, March 1956, pp. 421–514.

This is a review of theoretical work in the field of long distance telephone traffic. Because of the extensive use of special terminology used in telephone engineering and the emphasis on the engineering aspects, this paper better serves as an introduction for telephone engineers to the subject of stochastic models than as an introduction for mathematicians to the problem of telephone engineering. The only mathematical problem that is reviewed to any great extent is the following. Demand for telephone lines has a Poisson distribution and the durations of calls have an exponential distribution. Two types of service (telephone lines) are available, one with x parallel units and the other with y units. New calls are serviced by the x units whenever service facilities are free and by the y units only if the x units are all in use. The problem is to find the probability distribution for the number of y (overflow) units in use. *G. Newell.*

Melton, Ben S.; and Bailey, Leslie F. Multiple signal correlators. Geophysics **22** (1957), 565–588.

Methods of correlation of any number of input functions or signals to be compared are discussed. It is shown that correlation values can be obtained from any one of several arithmetic processes which measure the degree of similarity of the inputs, and that these processes can be mechanized. Included are a simple addition, a sign-coincidence scheme, and two processes based on analysis of variance. The discussion of analysis of variance procedures as applied to signals shows that the answers can be interpreted in terms of statistical significance, that measurements of coherent signal power can be made, and that the accuracy of such measurements is greatly improved as the number of input signals is increased. Signal-to-noise ratio is shown to be uniquely related to statistical significance, and signal detectability is plotted as a function of number of inputs and signal duration. Several means of presentation of data are suggested and illustrated. *Authors' summary.*

Törnebohm, Håkan. Kybernetik. Studium Gen. **10** (1957), 283–291.

See also: Probability: Watanabe. **Statistics:** Jensen; McGill.

Control Systems

Drenick, R. A non-linear prediction theory. Trans. I.R.E. **PGIT-4** (1954), 146–162.

The author considers the problem of optimal nonlinear prediction of a signal of the form

$$\bar{x}(t)=\theta_0+\theta_1 t+\cdots+\theta_q t^q \quad (0\leq t\leq 1),$$

where the θ_i are unknown constants, $\theta^r\neq 0$ and $q\leq k=$ given integer. The input to the predictor is assumed to be a linear mixture of $\bar{x}(t)$ and a stationary noise, $\epsilon(t)$, which is sampled by the predictor at the instants $t_i=i/n$ $(i=0, 1, 2, \cdots, n)$, n being a prescribed integer. The predictor is characterized by a relation of the form $x_p^*=f(x_0, x_1, \cdots, x_n)$, where $x_i=\bar{x}(-t_i)+\epsilon(-t_i)$, p is a positive integer and x_p^* is the predicted value of x_{-p} expressed as a function of the x_i $(i=0, 1, 2, \cdots, n)$. The loss function is assumed to be convex and twice differentiable at the origin.

A predictor is said to have the predictor property if $x_p^*=0$ for $x_0=x_1=\cdots=x_n=0$ and $x_p^*=\bar{x}_{-p}$ in the absence of noise. The author asserts without proof that the risk function of any predictor which has the predictor property is independent of the θ_i. Generalizing a method developed by Girshick and Savage [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of Calif. Press, 1951; pp. 53–73; MR **13**, 571] for the case $q=0$, the author obtains an implicit equation for the optimal predictor function, which reduces to an explicit equation in the case of a quadratic loss function, and illustrates the theory by several examples.

L. A. Zadeh (New York, N.Y.).

Kalman, R. E. Physical and mathematical mechanisms of instability in nonlinear automatic control systems. Trans. A.S.M.E. **79** (1957), 553–563, discussion 563–566.

The author gives in this paper a lucid discussion of the stability problem for control systems containing one or more nonlinear elements. It is written in a style which appeals equally to advanced design engineers and to research workers who deal with the theory of such systems.

The discussion is limited to those systems which can be treated as if they were made up of a finite number of "lumped" parts with time invariant parameters. Thus, such a system can be described by a finite number of simultaneous ordinary differential equations. The author exploits the topological approach to an analysis of nonlinear systems; i.e., he examines the nature of trajectories in phase space as generated by variations of inputs, variation of initial conditions, etc., and from these results deduces quantitative information about the possible existence of nodes, foci, and saddle points. During these analyses he makes use of the method of virtual critical points.

One of the main goals of the paper is to propose a classification of the types of nonlinearities which can give rise to instabilities. It is admitted that the proposed classification is tentative and that the accumulation of new knowledge may extend or modify the scheme proposed. By the use of carefully chosen but commonly encountered engineering situations, the author has been able to discuss instability in general terms as well as to arrive at five principal causes of instability: namely, (1) the root locus crosses over into the right half of the complex plane; (2) system equations are defined discontinuously; (3) multivalued nonlinearities; (4) the geometry of phase space; and (5) instability caused by sampling.

The paper is recommended reading for those persons interested in the theory of nonlinear control devices.

H. M. Trent (Washington, D.C.).

Freeman, E. A. The stabilization of remote position control systems by proportional Coulomb damping. J. Elektronics and Control (1) **3** (1957), 310–329.

A non-linear method of stabilizing remote position control systems is presented which has certain advantages over linear stabilization. The method is investigated analytically and by means of an electronic analogue computer. The characteristics of the transient in response to a a step function input are derived together with the stability criterion for any linear control system using this method of stabilization. *From the author's summary.*

Golay, Marcel J. E. The logic of bidirectional binary counters. I.R.E. Trans. **EC-6** (1957), 1–4.

The counters of Ware [Proc. I.R.E. **41** (1953), 1429–1437; MR **15**, 902] and Brown [I.R.E. Trans. **EC-4** (1955), 67–69] have the common property that no short-time internal memory is required for their correct operation. This property is achieved by the use of two toggles per counter stage. It is pointed out that Ware's counter is subject to dynamic instability during a part of the counting cycle, while Brown's counter never enters such a condition.

Brown's counter admits either of two counting directions depending upon phase relations between its two inputs. Eight modes of coupling are possible between counter stages of Brown's type; each coupling mode produces a different sequence of states during counting. These sequences are analyzed and tabulated.

D. E. Muller (Urbana, Ill.).

★ **Moisil, G. C. Sur la théorie algébrique des mécanismes automatiques: synthèse des schémas à relais polarisés.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 51–56. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

The author solves in detail a circuit problem involving a polarized relay, a three position relay controlled in its two active states by the direction of the energizing current in its winding. The problem is the determination of three Boolean functions of five Boolean variables by a recursive method. *D. H. Lehmer* (Berkeley, Calif.).

★ **Shestakov, V. I. Algebraic method of the synthesis of multicycle systems of r-position relays.** Translated by Morris K. Friedman, 572 California St., Newtonville 60, Mass., 1957. 6 pp.

Translated from Dokl. Akad. Nauk (N.S.) **112** (1957), 62–65.

See also: Ordinary Differential Equations: Babister; Eršov; Skatchkov.

HISTORY, BIOGRAPHY

★ **Menninger, Karl. Zahlwort und Ziffer. Eine Kulturgeschichte der Zahl. Bd. I. Zählreihe und Zahlsprache. 2. neubearbeitete und erweiterte Auflage.** Vandenhoeck and Ruprecht, Göttingen, 1957. 221 pp. DM 16.80.

The first edition of this book on the names and the symbols of numbers and the elementary operations of arithmetic appeared in 1934 in one volume [Hirt, Breslau]. The present edition appears in two volumes, contains much new material, embodies new understanding and has more pictures. This first volume deals with the language in which numbers are expressed, the second one will deal with the number symbols and with simple operations. Although the emphasis is on the origin of the indogermanic terms and in sepecial on the German expressions, ample space is given to finno-ugric, semitic and other language groups. The volume opens with the laws of formation of numbers after the concepts of "one" thing and of "two" things have been expressed, hence the creation of the terms for three and four, and then five with the terms obtained by composition and bundling. The units for bundling are usually ten, twenty, hundred. There are other methods of word formation, such as counting backwards: 19=20−1. The linguistic study of the terms themselves is preceded by a brief introduction into the basic features of the indogermanic languages, followed by that of some of the others. Central is the division of the indogermanic languages into the "kentum" and "satem" languages depending on the way they have adopted and transformed the original term "kṃtom" for hundred; this word itself is derived from "dékm(t)", ten, and means "Gezehnt" (collection of tens). Our "hundred" comes from gothic "hundra-rap", where "hunda" hails from "kṃtom" by means of well-known sound shifts, "rap" comes from gothic "rapjan" which means "to count" and is itself related to latin "ratio". The word "dékm(t)" probably is derived from "de-km(t)", two

hands. — This example may show the spirit of the book with its many details. Its rich content often yields surprises; it is not so easy to discover the "three" in "testament" (from latin "testis", witness, from "tri-sto", to stand in three), or in "trivial" (from latin "trivium", the commonplace subjects preceding the quadrivium). From Hungarian "husz"=20 comes "hussar", one cavalry man out of twenty houses. It is also instructive to read the course of cultural penetration from the wanderings of a numerical term: the French "quatre-vingt", replacing the Latin decimal form (perhaps "huitante", or "octante") in the eleventh century, beginning in Northern France, shows the influence of the Norman penetration — the vigesimal system being usual in the nordic regions. And, from whatever country the gypsies =(e)gypsies may have come, it is not Egypt, but from a country with an indogermanic language, as their number system shows.

D. J. Struik (Cambridge, Mass.).

★ **Gauss, C. F. Gedenkband anlässlich des 100. Todestages am 23. Februar 1955.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. iii+251 pp. $4.80.

An elegantly bound and printed volume with eleven essays by German and Russian mathematicians on the various aspects of Gauss' life and work. There are two portraits of Gauss and a map illustrating his activity in geodesy.

Rumyancev, V. V. The great Russian savant A. M. Lyapunov. (On the 100th anniversary of his birth). Vestnik Akad. Nauk SSSR (1957), no. 6, 44–49. (Russian)

A description in popular language of the scientific achievements of Lyapunov.

Galli, Mario. Semplici considerazioni sopra un presunto errore di Galileo. Boll. Un. Mat. Ital. (3) **12** (1957), 80–82.

A passage in the second day of the dialogue Massimi Sistemi, often thought to be an error on Galileo's part, is here shown to be merely an argumentum ad hominem directed against the peripatetic Simplicio.

Smirnov, V. I. Leonard Euler (on the 250th anniversary of his birth). Vestnik Akad. Nauk SSSR **27** (1957), no. 3, 61–68. (Russian)

The essay is in three sections; the first gives general biographical details, the second a general description of Euler's scientific works, and the third a detailed description in popular language of his chief contributions to mathematics, physics and astronomy.

★ **Hofmann, Joseph Ehrenfried. Geschichte der Mathematik. Zweiter Teil. Von Fermat und Descartes bis zur Erfindung des Calculus und bis zum Ausbau der neuen Methoden.** Sammlung Göschen Bd. 875. Walter de Gruyter and Co., Berlin, 1957. 109 pp. DM 2.40.

★ **Hofmann, Joseph Ehrenfried. Geschichte der Mathematik. Dritter Teil. Von den Auseinandersetzungen um den Calculus bis zur Französischen Revolution.** Sammlung Göschen Bd. 882. Walter de Gruyter and Co., Berlin, 1957. 107 pp. DM 2.40.

[For the first volume (1953) of these booklets see MR **15**, 275.] The second and third volumes carry us from what is called "High Baroque" (c. 1625–c. 1665) via the "Late Baroque" (c. 1655–c. 1730) into the Enlightenment (c. 1700–c. 1790). The second volume concentrates on the development of the calculus of infinitesimals from the time of Descartes and Fermat via Torricelli, Huygens Wallis, Mengoli, Gregory and others to the period immediately after the fundamental discoveries of Newton and Leibniz. Here the author gives us the benefit of his minute knowledge and understanding of the literature of this period; his account almost runs from year to year and we can follow even in the compact form of the narrative the human tensions present in the successive discoveries. This story is continued in the third volume with the priority struggles between the protagonists and their epigones, and then passes to an account of the works of the leading authors of the eighteenth century: the Bernoullis, Euler, Lambert, Lagrange, Monge, Legendre, and the early work of Laplace. The Japanese mathematicians of this period are not neglected. Room has also been found for the majestic unfolding of the calculus, but also for many contributions in the fields of algebra, number theory and geometry. An outstanding feature of these volumes is the extensive bio-bibliography, to which the text, especially in the third volume, appears occasionally as a running commentary. No worker in the history of mathematics can do without these little books, with their enormous amount of information in condensed form, and their outlook into the general field of culture.

D. J. Struik (Cambridge, Mass.).

Рыбкин, Г. Ф.; и Юшкевич, А. П. (Редакторы). Историко-математические исследования. Выпуск IX. [Rybkin, G. F.; and Yuškevič, A. P. (Editors). Historico-mathematical investigation. Issue IX.] Gosudarstv. Izd. Tehn.-Teor. Lit., Moscow, 1956. 803 pp. 23.05 rubles.

This issue contains four types of contributions: (1) very detailed information about documents concerning the date and place of birth of N. I. Lobačevskiĭ; (2) articles about the spreading and development of Lobačevskiĭ's ideas, including biographies by professors F. M. Suvarov and A. P. Kotel'nikov; (3) eight papers on the history of Mathematics in the Ukraine, and (4) two papers not related to the preceding, one on the treatise of Archimedes on Floating Bodies by I. G. Bašmakov, and one on the investigations of Dalambert and Euler in the Theory of Linear Systems of Differential Equations with Constant Coefficients by N. I. Simonov. *G. Y. Rainich.*

Varoli, G. Obituary: Filippo Sibirani. Boll. Un. Mat. Ital. (3) **12** (1957), 125–130.

A short scientific biography.

Duffin, R. J. Obituary: A. C. Schaeffer, mathematician. Science **126** (1957), 156.

★ **Bianchi, Luigi. Opere. Vol. VI. Congruenze di rette e di sfere e loro deformazioni.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1957. 328 pp. 3000 Lire.

Twelve articles, with a short introduction by Vittorio Strazzeri.

Borkowski, Ludwik. First modern monograph on Aristotle's syllogistic. Studia Logica **5** (1957), 13–26. (Polish)

An expository discussion of the book by Jan Łukasiewicz reviewed in MR **14**, 713.

Mel'nikov, I. G. Leonard Euler and elementary mathematics. Mat. v. Škole 1957, no. 4, 1–15. (Russian)

Essays on Euler's work in the theory of numbers, and in elementary algebra, geometry and trigonometry, and on his influence as a creator of modern mathematical language.

Rychlík, Karel. Theory of real numbers in the manuscripts left by Bolzano. Časopis Pěst. Mat. 81 (1956), 391–395. (Czech)

In the writings of Bolzano so far published there appear many theorems about real numbers, particularly in his first works on analysis, "Der binomische Lehrsatz ··· (1816)" and "Rein analytischer Beweis ··· (1817)", and also in "Functionenlehre" published in 1930 long after Bolzano's death. In the "Functionlehre" there are references to various passages in still unpublished manuscripts. In the manuscript "Unendliche Zahlen-(Grössen)-begriffe (Zahlenlehre II)" the theory of real numbers is systematically worked out. In the present article the author reports on this manuscript. Many of the concept used in the "Functionlehre" are explained in it, and the source of several theorems that were used in the "Functionenlehre" without proof is cleared up. *Author's summary.*

★**Sarton, George. The study of the history of mathematics and the study of the history of science.** (Two volumes bound as one.) Dover Publications, Inc., New York, N. Y., 1957. 113 pp.+75 pp. $1.25.

A paperbound reprint of the two works published originally in 1936 by the Harvard University Press, Cambridge, Mass.

Rosenfeld, L. The velocity of light and the evolution of electrodynamics. Nuovo Cimento (10) 5 (1956), supplemento, 1630–1669.

A vigorous, documented, philosophico-historical account of the evolution of electromagnetic theory from Weber to Hertz. *J. L. Synge* (Dublin).

Obituary: Anonymous. Jan Łukasiewicz (1878-1956). Studia Logica 5 (1957), 7–11. (Polish)

An obituary notice with bibliography of 90 entries.

Tricomi, Francesco G. Necrologio: Guido Ascoli. Boll. Un. Mat. Ital. (3) 12 (1957), 347–350.

A short general and scientific biography with a photograph.

MISCELLANEOUS

★**Gaines, Helen Fouché. Cryptanalysis, a study of ciphers and their solution.** Dover Publications, Inc., New York, 1956. vi+237 pp. Paperbound: $1.95; clothbound: $3.95.

This is a reprinting of the American Photographic Publishing Company's [Boston] "Elementary cryptanalysis" first published in 1939. There is internal evidence in this book that the author approached the subjects as a puzzle solver, the 166 problems in the book being mostly from members of the American Cryptogram Association. However, the book goes beyond the simple substitutions used for most puzzles. It takes up Vigenère and other examples of multiple alphabet substitutions with fixed periods, some of which have been used for military and diplomatic purposes. Methods of analysis applicable to periodic ciphers in general are discussed. There is a page devoted to "running key" and a whole chapter to "autoencipherment." There is a chapter on the diagraphic Playfair system which was used by the Allies in World War I, and a chapter on fractionating systems in which the basic elements are not letters but something smaller, a fraction of a letter. The algebraic cipher of Lester S. Hill, which might be of interest to readers of these Reviews, gets only a brief mention. The study of ciphers avoids entirely the subjects of codes, in which the basic units are not letters but words or even phrases.

All in all this is an excellent book, very well founded in the basic elements of cryptanalysis, as is shown by the fact that its stature has diminished hardly at all since its first publication seventeen years ago. This was an extraordinary accomplishment by Mrs. Gaines in a field in which secretiveness and deception are basic principles.

H. Campaigne (Washington, D.C.).

★**Oxford Mathematical Conference, abbreviated proceedings.** Trinity College, Oxford, April 8–18, 1957. Technology, The Times Publishing Company Ltd., London, 1957. 111 pp. 2s. 6d.

A conference of schoolteachers and industrialists with 28 articles on mathematics in industry and the corresponding changes that should be made in teaching in secondary schools. Certain other contributions, to be published elsewhere, are mentioned by title only. The conference was supported by 18 industrial and government organizations. As a "rather unrepresentative" example of the use of computers the value of π is given to 10,000 decimal places.

Ważewski, T. Influence of new mathematical methods on the development of classical disciplines of mathematics. Prace Mat. 2 (1956), 1–26. (Polish)

★**Shamos, Morris H.; and Murphy, George M. (Editors). Recent advances in science: Physics and applied mathematics.** New York University Press, New York. Distributors: Interscience Publishers, New York-London, 1956. xi+384 pp. $7.50.

Chapter One, entitled Methods of applied mathematics, is by R. Courant and Chapter Two, The future of operations research, is by P. M. Morse, while the other ten chapters are on physical subjects. The book, written in semi-popular style, is an outgrowth of the First Symposium on Recent Advances in Science, held in New York in 1954 under the auspices of the Division of General Education of New York University.

★**Peltier, Jean. Inventaire collectif des périodiques mathématiques. I. Conventions, notations et répertoire par mots-types, à la date de juin 1956.** Documentation mathématique, Fasc. 36. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1957. 53 pp. (polycopiées) 1200 francs.

★**Peltier, Jean. Inventaire collectif des périodiques mathématiques. II. Catalogue alphabétique par titres donnant l'état des collections de 13 bibliothèques parisiennes, à la date de juin 1956.** Documentation mathématique, Fasc. 37. Secrétariat Mathématique, 11 rue Pierre Curie, Paris, 1957. 159 pp. (polycopiées). 1200 francs.

★**Gouyon, R. Précis de mathématiques spéciales.** Librairie Vuibert, Paris, 1956. ix+647 pp. 6300 francs.

This work is addressed to the "classe de mathématiques spéciales (programme A_2)" in French schools, and also to the "classe de mathématiques supérieures (programme A_1)". The style is concise, particularly in the proofs and in the exercises, so that a great deal of information can be included. There are five sections: notions fondamentales, compléments d'algèbre et applications géometriques, éléments de calcul différentiel, coniques et quadriques, éléments de calcul intégral, éléments de mécanique, with a complementary note on descriptive geometry.

Andronov, I. K. Results of forty years of development in mathematical education in the USSR. Mat. v Škole 1957, no. 5, 6–21. (Russian)

A great deal of qualitative and quantitative information, largely with respect to secondary schools.

Belyĭ, B. N. Sketch of the development of mathematical methodology in the Ukraine in 40 years. Mat. v Škole 1957, no. 5, 22–39. (Russian)

A study of mathematical pedagogy, divided into 3 periods: 1917–1932 (during this period the school system of the Ukraine was different from that of the RSFSR), 1933–1941, and 1945–1957. There is a bibliography with 166 entries.

★**Александров, А. Д.; Колмогоров, А. Н.; и Лаврентьев, М. А. [Aleksandrov, A. D.; Kolmogorov, A. N.; i Lavrent'ev, M. A. (Editors).] Математика: ее содержание, методы и значение. [Mathematics: its content, methods and meaning.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. Vol. 1, 296 pp. 15.85 rubles; Vol. 2, 395 pp. 20.80 rubles; Vol. 3, 336 pp. 17.60 rubles.

An example of popularization of the best kind. There are twenty chapters written by about the same number of outstanding Russian mathematicians. Algebra, geometry, analysis and topology are well represented and there are chapters on number theory, numerical analysis and computing machines. There is very little on foundations or logic. High school mathematics is sufficient preparation for the earlier chapters but somewhat more is needed later.

AUTHOR INDEX

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